WORKSHEET #13 – MATH 1260
FALL 2014

NOT DUE

1. Short answer:

(a) Find the equation of the tangent plane to $z = x^2 + y^2$ at the point $\langle 1, 1, 2\rangle$.

(b) Find the equation of the tangent line to $\vec{r}(t) = \langle t, t^2, t+1\rangle$ at the point $\langle 1, 1, 2\rangle$.

(c) Compute the curvature of $\vec{r}(t) = \langle t, t^2, t+1\rangle$ at $\langle 1, 1, 2\rangle$.

(d) What is the distance between the plane $x + 3y - 4z = 2$ and the point $\langle 1, 1, 1\rangle$.

(e) Suppose the temperature of a hot plate at position $\langle x, y\rangle$ is given by $z = ye^{xy} + y^2 + 2x + 120$. If an intrepid ant is at the origin, what direction should the ant move in order to lower the temperature on his feet?

(f) Give an example of 3 vectors in $\mathbb{R}^3$ that do not form a basis for $\mathbb{R}^3$.

2. Short answer:

(a) Give an example of a function $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ that is not one-to-one.

(b) Setup an integral that would compute the arclength of the curve $\vec{r}(t) = \langle t^2, t^3, t^4 \rangle$ between $t = 1$ and $t = 3$.

(c) Give an example of a vector field on $\mathbb{R}^3$ that is not the gradient of a function $w = f(x, y, z)$.

(d) Is the following vector field the curl of another vector field? $\vec{F} = x^2\vec{\imath} + (z^3 - 2yx)\vec{\jmath} + x\vec{k}$.

(e) Consider the surface S parameterized by $r(u, v) = \langle u^2, e^{uv}, v^2 \rangle$. Setup an integral to compute the surface area of S over the region where $0 \le u \le 1$ and $-1 \le v \le 1$.

(f) Suppose that $\vec{F}$ is a conservative vector field on $\mathbb{R}^2$. Let C_1 be the curve parameterized by $\vec{u}(t) = \langle t^2, 2t \rangle$ for $0 \le t \le 1$ and let C_2 be the curve parameterized by $\vec{v}(t) = \langle \sin((\pi/4)t), 3^{t/2} - 1 \rangle$ for $0 \le t \le 2$. Is

$$\int_{C_1} \vec{F} \cdot ds = \int_{C_2} \vec{F} \cdot ds?$$

3. Short answer:

(a) Give an intuitive description of what the inverse function theorem says. Include pictures.

(b) Precisely state the implicit function theorem.

(c) Give an example of a subset of $\mathbb{R}^2$ that is neither open nor closed.

(d) Is the set of integers (whole numbers) a compact subset of $\mathbb{R}$?

(e) What does Green's theorem say?

(f) Give an example of a function $\mathbb{R}^3 \longrightarrow \mathbb{R}$ that is onto.

(g) Is the following parameterization of a curve a parameterization with respect to arclength?

$$\vec{r}(t) = \frac{1}{\sqrt{2}}\langle \cos(t), \sin(t), t\rangle$$

4. Find the distance between the plane $x + 2y + 3z = 0$ and the point $\langle 4, 3, 2 \rangle$.

5. Consider the vector field $\vec{F} = \langle -y, x \rangle$. Find a line L and a parametric representation $\vec{r}'(t)$ of it so that the tangent vector $\vec{r}'$ of the line at any point $Q = \langle a, b \rangle$ has a multiple equal to $\vec{F}(\langle a, b \rangle)$. Bonus points if you can parameterize L in a non-linear way so that $\vec{r}'(Q) = \vec{F}(Q)$.

6. Find the distance of the point $(1, 2, 3)$ from the tangent plane to the surface parameterized by $\vec{r}(u, v) = \langle u^2, uv, v^3 \rangle$ at $(1, 11)$.

7. Recall that the transformation for rotation by θ degrees is given by $\vec{r}(u, v) = \langle u\cos(\theta) + v\sin(\theta), -u\sin(\theta) + v\cos(\theta)\rangle$. Use this to deduce the formula for $\sin(\alpha + \beta)$.

8. The base of an aquarium of volume V is made of stone and the sides are glass (there is no top). If stone costs 8 times as much as glass, what dimensions should the aquarium be (in terms of V) in order to minimze the cost of materials. Justify your answer. (You can use whatever method you want).

9. Reparameterize the curve $t \mapsto \langle \cos(2\pi t), \sin(2\pi t), t\rangle$ with respect to arc length.

10. Consider the integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} dz\,dy\,dx$$

it gives the volume of some region. Draw the region, and then setup the same integral both polar and in spherical coordinates. (You don't need to evaluate the integral, unless you want to).

11. Consider the region above the plane $z=-2$, below the plane $z=4y$ and inside the cylinder $x^2+y^2=1$ (in other words, $z\geq -2, z\leq 4y, x^2+y^1\leq 1$). Draw the region. Suppose now that the density of the object is given by the formula $\rho(x,y,z)=5x^2e^yz$. Setup, but do not evaluate an integral that computes the mass of the region.

12. Consider a particle moving along a curve from $(0,1)$ to $(1,0)$ via the parameterization $\vec{r}(t) = \langle t^2, e^{t(1-t)}(1-t)^{49}\rangle$ for $t = 0$ to 1. Find the work done by the force field $\vec{F} = y^2\vec{i} + 2xy\vec{j}$ as the particle moves along this curve segment.

13. Consider the wire parameterized by the formula $\vec{r}(t) = \langle t\cos(t), \sin(t)\rangle$, for $t = 0$ to $t = 2\pi$. Suppose the mass of the wire per unit length at point (x, y) is given by $\mu(x, y) = x^2 + y^2 + 1$. Setup an integral to compute the mass of the wire.

14. Let S be the surface defined by the equation

$$z = x(1-x)y(1-y)(7 + \sin(xy) + e^{\cos(x)})$$

and above the square $0 \leq x \leq 1$, $0 \leq y \leq 1$. Let $\vec{F}$ be the vector field $y^2\vec{\imath} - \vec{\jmath} + x\vec{k}$. Compute the flux integral

$$\iint_S \vec{F} \cdot dS.$$

15. Let S_1 be the surface defined by $z = 0, x \geq 0, y \geq 0$, and $x + y \leq 1$. Let S_2 be the surface defined by $z = e^{\cos(x)}x^4y^2(1-x-y)^3, z \geq 0, x \geq 0, y \geq 0$. Suppose that the volume of the region below S_2 and above S_1 is K. If $\vec{F} = \langle \frac{1}{K}x, z, 2y \rangle$, compute

$$\iint_{S_2} \vec{F} \cdot dS.$$

16. State and prove the Heine-Borel theorem. You may (and should) use the following fact.

Fact: If $B \subseteq \mathbb{R}$ is a bounded nonempty set, then B has a least upper bound. In other words there is a number $L > 0$ so that $L \geq b$ for all $b \in B$ (this just says that L is an upper bound for b) and such that if M is any other upper bound for B, then $L \leq M$ (this just says that L is $\leq$ any other upper bound).

17. Suppose that $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a continuous function. Prove that if $W \subseteq \mathbb{R}^n$ is a compact set, then $f(W) \subseteq \mathbb{R}^m$ is also compact.

Hint: Let $\{U_i\}$ be an open cover of $f(W)$ (presumably there are infinitely many U_i). You need to prove that there is a finite subcover of the U_i's.