WORKSHEET #10 – MATH 1260
FALL 2014

NOT DUE, NOVEMBER 21ST

1. First we begin with short answer questions.

(a) Define the term *open cover*.

Solution: An open cover a set $W \subseteq \mathbb{R}^n$ is a collection of open sets $\{U_i\}$ such that every point of W is in at least one U_i.

(b) Define the term *compact*.

Solution: A subset $W \subseteq \mathbb{R}^n$ is called *compact* if every open cover of W has a finite subcover.

(c) Write down an open cover of the open interval $(0, 1)$ that does not have a finite subcover.

Solution: For each integer $i \geq$ consider $U_i = (\frac{1}{i} - \frac{1}{2i}, \frac{1}{i} + \frac{1}{2i})$. These ought to do the job. There is no finite cover because a finite number of them cannot get arbitrarily close to zero.

(d) Write down a precise definition of what it means for a function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ to be continuous.

Solution: You could simply say the inverse image of an open set is open, or you could say for every $a \in \mathbb{R}^n$, and every $\varepsilon > 0$, if $B_{f(a),\varepsilon}$ is a ball of radius ε around $f(a) \in \mathbb{R}^m$ then there is a $\delta > 0$ so that if $B_{a,\delta}$ is a ball of radius δ around a, then $f(B_{a,\delta}) \subseteq B_{f(a),\varepsilon}$.

(e) If $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$, for $t = 0$ to 3, is a parameterization of a curve in $\mathbb{R}^3$, write down an integral that would compute its arclength.

Solution:

$$\int_0^3 |r'(t)|dt \text{ or } \int_0^3 \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt$$

would work fine.

(f) State Green's theorem.

Solution: If $\vec{F} = \langle P, Q\rangle$ is a vector field on $\mathbb{R}^2$ with continuous partial derivatives and $A \subseteq \mathbb{R}^2$ is a closed path connected region bounded by a simple piece-wise smooth closed curve C, then

$$\int_C \vec{F} \cdot ds = \iint_A (Q_x - P_y)dA.$$

(g) State the divergence theorem.

Solution: If $\vec{F} = \langle P, Q, R\rangle$ is a vector field on $\mathbb{R}^3$ with continuous partial derivatives and $A \subseteq \mathbb{R}^3$ is a closed path connected 3-dimensional region bounded by a piece-wise smooth surface S (with outward orientation) then

$$\int\int_S \vec{F} \cdot dS = \int\int\int_A \text{div}\vec{F} dV.$$

We continue **1.**

(h) Is the following vector field the gradient of a potential function? $\vec{F} = \langle y, y, \cos(z) \rangle$

Solution: The curl is nonzero, so no it is not.

(i) Rewrite the integral

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{0}^{\sqrt{1-x^2-y^2}} x^2 y dz dy dx$$

in spherical coordinates (do not evaluate or simplify unless you want to).

Solution: This is just the top half of a sphere. So we integrate

$$\int_0^{2\pi} \int_0^{\pi/2} \int_0^1 (r\cos(\theta)\sin(\phi))^2 (r\sin(\theta)\sin(\phi))(r^2\sin(\phi)) dr d\phi d\theta$$

(j) Is the following vector field the curl of some other vector field $F = \langle y e^{\cos(z)}, x^2 \frac{1}{1+x^2+z^2}, e^x * y \rangle$?

Solution: The div is zero, so yes it is.

(k) Use Green's theorem to compute the work done by the vector field $\vec{F} = \langle y, 2x \rangle$ on a particle that moves in a circle of radius 2 around the point $\langle 23, 111111 \rangle$.

Solution: $Q_x - P_y = 1$, so we are just integrating this over the circle of radius 1 centered at the point $\langle 23, 111111 \rangle$. But that's just computing the area of the circle which is of course equal to $\pi r^2 = 4\pi$.

(l) Give an example of a vector field that is not conservative.

Solution: $\vec{F} = \langle y, 0 \rangle$ works since $Q_x - P_y \neq 0$.

(m) Setup an integral to compute the volume of a region below the $z = x + y + 10$ plane, above the paraboloid $z = -5 + x^2 + y^2$ and inside the cylinder $x^2 + y^2 = 1$. Do not evaluate the integral.

Solution:

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-5+x^2+y^2}^{x+y+10} 1 dz dy dx$$

One can do it in cylindrical coordinates too of course.

(n) Parameterize the part of the paraboloid $z = x^2 + y^2$ that lies above the triangle with vertices $\langle 0,0,0 \rangle, \langle 0,1,0 \rangle, \langle 1,0,0 \rangle$. Use a function $\vec{r}(u,v)$ and make sure to specify the domain that the u, v are allowed to come from.

Solution: We parameterize the parabaloid by $\vec{r}(u,v) = \langle u, v, u^2 + v^2 \rangle$. We have $u \geq 0$, $u \leq 1 - v$, $v \in [0,1]$.

2. Consider a particle moving along a curve parameterized by $\vec{r}(t) = \langle t(t-1)e^{\sin(t)} + t^2, 1 - t - (e^t - 1)(t-1)\sin(e^t)\rangle$ for $t = 0$ to 1. Find the work done by the force field $\vec{F} = 2xy\vec{i} + (x^2+1)\vec{j}$ as the particle moves along this curve segment.

Solution: The curve is really messy but notice that the vector field $\vec{F}$ is conservative. Indeed $\vec{F} = \nabla x^2 y + y = \nabla f$. Notice also that $\vec{r}(0) = \langle 0, 1\rangle$ and that $\vec{r}(1) = \langle 1, 0\rangle$. Hence by the fundamental theorem for line integrals, we only have to compute $f(\vec{r}(1)) - f(\vec{r}(0)) = f(1,0) - f(0,1) = 0 - 1 = -1$. Alternately, you could have just done a line integral along the line from $\langle 0, 1\rangle$ to $\langle 1, 0\rangle$.

3. Consider the region above the plane $z = -1$, below the plane $z = 2x$ and inside the cylinder $x^2 + y^2 = 1$ (in other words, $z \geq -1, z \leq x, x^2 + y^1 \leq 1$). Draw the region. Suppose now that the density of the object is given by the formula $\rho(x,y,z) = 5\cos(x)e^z$. Setup, but do not evaluate an integral that computes the mass of the region.

Solution: First we have to figure out where the two planes intersect. $z = -1$ and $z = 2x$ means that $2x = -1$ so $x = -\frac{1}{2}$. So since we need $2x \geq -1$, we really must have $x \geq -\frac{1}{2}$. The following integral will work

$$\int_{-\frac{1}{2}}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-1}^{2x} (5\cos(x)e^z)\,dzdydx$$

6. Let S be the surface defined by the equation

$$z = x(1-x)y(1-y)(7 + \sin(xy) + e^{\cos(x)})$$

and above the square $0 \le x \le 1$, $0 \le y \le 1$. Let $\vec{F}$ be the vector field $y^2\vec{i} - \vec{j} + x\vec{k}$. Compute the flux integral

$$\iint_S \vec{F} \cdot dS.$$

Solution: One quickly notices that $\div(\vec{F}) = 0$ so this flux integral is independent of surface. Now one could figure out what this vector field is the curl of and do a line integral (with 4 components) it's just as easy to notice that this integral is independent of surface. So let's parameterize the surface $\vec{r}(u,v) = \langle u, v, 0\rangle$ and notice it has the same boundary as S. We notice that the normal vector with this parameterization is $\langle 0,0,1\rangle$ (you might even be able to see this without a cross product...). Then we compute the flux integral

$$\int_0^1 \int_0^1 \langle v^2, -1, u\rangle \cdot \langle 0,0,1\rangle du dv = \int_0^1 \int_0^1 u du dv = \int_0^1 u^2/2|_0^1 dv = \int_0^1 \frac{1}{2} dv = \frac{1}{2}.$$

7. Suppose an extremely complicated surface S (with upward orientation) defined by a (continuously differentiable) equation $z = f(x,y)$ is always ≥ 2 and intersects the $z = 2$ plane in a circle of radius 2 centered at the point $\langle 0,0,2\rangle$. Further suppose that the volume of the region above the plane $z = 2$, below the surface $z = f(x,y)$ and above the aforementioned circle is equal to 3. Compute

$$\iint_S \langle 3x, e^z, y+1\rangle \cdot dS.$$

Solution: We notice that $\text{div}\langle x, e^z, y\rangle = 3$. We also observe that if S_2 is the surface which is the circle of radius 2 centered at $\langle 0,0,2\rangle$ with upwards orientation then

$$-\iint_{S_2} \langle 3x, e^z, y+1\rangle \cdot dS + \iint_S \langle 3x, e^z, y+1\rangle \cdot dS = \iiint_E \text{div}\langle 3x, e^z, y+1\rangle dV \iiint_E 3dV$$

where E is the region bounded by S and S_2 and so the above is equal to $3 \cdot 3 = 9$. Then we try to solve for $\iint_S \langle 3x, e^z, y+1\rangle \cdot dS$. Thus we just need to compute $\iint_{S_2} \langle 3x, e^z, y+1\rangle \cdot dS$. We parameterize S_2 by $\vec{r}(u,v) = \langle u, v, 2\rangle$. Clearly the normal vector (with upward orientation) is just $\langle 0,0,1\rangle$. Hence we compute

$$\int_{-2}^2 \int_{-\sqrt{4-v^2}}^{\sqrt{4-v^2}} (v+1) du dv.$$

We turn this into polar coordinates $(u = r\cos(\theta), v = r\sin(\theta))$.

$$\int_0^{2\pi} \int_0^2 (r\sin(\theta) + 1) r dr d\theta = \int_0^{2\pi} (\sin(\theta)\frac{8}{3} + 2) d\theta = \frac{8}{3}(-\cos(\theta) + 2\theta)|_0^{2\pi} = 4\pi.$$

Hence our desired integral is equal to $9 + 4\pi$.