WORKSHEET #1 – MATH 1260
FALL 2014

DUE TUESDAY, SEPTEMBER 2ND

For this assignment, you are allowed and encouraged to work in groups. Each group only has to turn in one assignment worksheet, but make sure it is done neatly.

We'll learn some things about vectors that are not in the book.

Suppose $\vec{v}_1, \ldots, \vec{v}_m$ are in $\mathbb{R}^n$. We say that these vectors *span* $\mathbb{R}^n$ if every vector $\vec{w}$ in $\mathbb{R}^n$ can be written as

$$\vec{w} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m.$$

for some real numbers $a_i \in \mathbb{R}$ (such a sum with real coefficients is called a *linear combination*). Geometrically, this means that the vectors don't all lie in the same hyperplane. (So in $\mathbb{R}^3$, it means they don't all lie in a plane).

1. Show that $\vec{i} = \langle 1, 0\rangle, \vec{j} = \langle 0, 1\rangle$ span $\mathbb{R}^2$.

Hint: Given $\vec{u} = \langle a, b\rangle$, how do you write it as a linear combination of $\vec{i}$ and $\vec{j}$?

2. Do $\vec{i}, \vec{j},$ and $\vec{w} = \langle 3, 2, -1\rangle$ span $\mathbb{R}^3$? If so, prove it (give a justification).

3. Show that $\vec{u} = \langle 1, 0, 1\rangle, \vec{v} = \langle 2, 1, 0\rangle,$ and $\vec{w} = \langle -1, -1, 1\rangle$ do not span $\mathbb{R}^3$.

Hint: Two vectors that are not collinear obviously span a plane. Find a vector that cannot be written as a linear combination of the three vectors above.

4. Show that $\vec{i}, \vec{j}, \vec{k},$ and $\vec{w} = \langle 1, 2, 3 \rangle$ span $\mathbb{R}^3$.

Another way to say that $\vec{v}_1, \ldots, \vec{v}_m$ span $\mathbb{R}^n$ is to say that every other vector $\vec{u} \in \mathbb{R}^n$ can be written as a linear combination of the $\vec{v}_i$ in *at least one way*. This leads us to our next notion.

We say that $\vec{v}_1, \ldots, \vec{v}_m$ are *linearly independent* in $\mathbb{R}^n$ if for each vector $\vec{u} \in \mathbb{R}^n$, there exists *at most one* linear combination of the $\vec{v}_i$ that equals $\vec{u}$. In other words, if there are at most one set of real numbers $a_1, \ldots, a_m \in \mathbb{R}^n$ so that

$$\vec{u} = a_1\vec{v}_1 + \ldots + a_m\vec{v}_m.$$

By the way, sets of vectors that are not linear independent are called *linearly dependent*.

5. Show that $\vec{i}, \vec{j}, \vec{k}$ form a linearly independent set of vectors in $\mathbb{R}^3$.

Hint: Suppose that we can write

$$a_1\vec{i} + b_1\vec{j} + c_1\vec{k} = a_2\vec{i} + b_2\vec{j} + c_2\vec{k}.$$

Then write each side as a single vector $\langle \ldots, \ldots, \ldots, \rangle$ and deduce that $a_1 = a_2, b_1 = b_2, c_1 = c_2$. Why is doing this enough?

6. Find two linearly dependent vectors in $\mathbb{R}^2$. Do they span $\mathbb{R}^2$?

7. Find three linearly dependent vectors in $\mathbb{R}^2$ that span $\mathbb{R}^2$.

8. Find three different linearly dependent vectors in $\mathbb{R}^3$.

9. Are the vectors $\vec{x} = \langle 1, 0, 0, 1\rangle, \vec{y} = \langle , 0, 1, 1, 0\rangle, \vec{z} = \langle 0, 0, 2, 1\rangle, \vec{w} = \langle 0, 0, 0, -1\rangle$ linearly independent in $\mathbb{R}^4$? Do they span $\mathbb{R}^4$?

A linearly independent spanning set is called a *basis*. Here are some facts about bases. You may take these as given going forward.

(1) Any basis in $\mathbb{R}^n$ has exactly n elements.
(2) Any linearly independent set of n vectors in $\mathbb{R}^n$ is automatically a spanning set (and hence a basis).
(3) Any spanning set of n vectors in $\mathbb{R}^n$ is automatically linearly independent (and hence a basis).

For extra credit, up to 6 homework points, each individual must write a 1-2 page justification of the facts above (to learn more, find any text on Linear Algebra).