HARD HOMEWORK #2 – MATH 1260
FALL 2014

DUE, MONDAY DECEMBER 8TH

This homework is on the more theoretical aspects of Calculus that we have been discussing. Of course, I encourage you to work in groups, but each person needs their own write-up.

Let us recall some definitions.

Definition (injective) A function $\vec{f} : U \longrightarrow V$ (with $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$) is called *injective* (or *one-to-one*) if whenever $\vec{f}(\vec{u}_1) = \vec{f}(\vec{u}_2)$ then $\vec{u}_1 = \vec{u}_2$ (for any $\vec{u}_1, \vec{u}_2 \in U$). Loosely speaking, this means each potential output is hit at most once.

Definition (surjective) A function $\vec{f} : U \longrightarrow V$ (with $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$) is called *surjective* (or *onto*) if for every $\vec{y} \in V$, there exists a $\vec{x} \in U$ with $\vec{f}(\vec{x}) = \vec{y}$. Loosely speaking, this means each potential output is hit at least once.

Definition (dense) Suppose that $W \subseteq A \subseteq \mathbb{R}^n$ are subsets with $A \subseteq \mathbb{R}^n$ open. We say that W is *dense* in A if every open non-empty subset of A also contains some points of W. For example, the rational numbers are *dense* in the real numbers since every open interval contains some real numbers.

1. Suppose that $\vec{f} : U \longrightarrow V$ and $\vec{g} : V \longrightarrow W$ are functions with $U \subseteq \mathbb{R}^n$, $V \subseteq \mathbb{R}^m$ and $W \subseteq \mathbb{R}^o$. Suppose that $\vec{g} \circ \vec{f}$ is surjective. Show that $\vec{g}$ is then surjective but that $\vec{f}$ isn't necessarily surjective.

2. Suppose that $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation given by a matrix[1] $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Suppose that A has non-zero determinant, find an inverse linear transformation for T, and show that T is both injective and surjective.

3. Find a function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ and another function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ such that $g \circ f$ is the identity on $\mathbb{R}$ but such that $f \circ g$ is *not* the identity on $\mathbb{R}^2$.

4. Suppose $A \subseteq \mathbb{R}^n$ is an open set and that $\vec{f} : A \longrightarrow \mathbb{R}^n$ is continuously differentiable and injective. Further suppose that the Jacobian matrix of f always has non-zero determinant (for any point $\vec{x} \in A$). Show that $f(A)$ is open. Give an example which shows that $f(A)$ is not necessarily open if the Jacobian matrix of f has zero determinant.

5. Suppose that $\vec{f} : A \longrightarrow \mathbb{R}$ is a continuous function for some open set $A \subseteq \mathbb{R}^n$. Suppose that there is a dense subset of A, $W \subseteq A$ with $\vec{f}(\vec{w}) = 0$ for every $\vec{w} \in W$. Show that $\vec{f}$ is the constant zero function.

[1] meaning that $T(\langle x_1, x_2 \rangle) = \langle ax_1 + bx_2, cx_1 + dx_2 \rangle$

Hint: Suppose that $\vec{f}(\vec{x}) \neq 0$ for some $\vec{x} \in A$. Use the continuity hypothesis to show that there is an open set around $\vec{x}$ all of whose elements are also not sent to zero. Then use this to contradict our density assumption on W.

6. Suppose that $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function with continuous partial derivatives. Show that f is not injective.[2]

Hint: First show that there must be places where the partial derivative f_x is zero (and likewise with f_y) on every open subset $A \subseteq \mathbb{R}^2$. Indeed if it is never zero, consider the function $\vec{g} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $\langle x, y\rangle \mapsto \langle f(x, y), y\rangle$. Compute the Jacobian of this function and conclude that $\vec{g}(A)$ is open in $\mathbb{R}^2$ by using an earlier exercise. Next conclude that this means that the places where f_x are zero are a dense subset of $\mathbb{R}^2$ and hence that f_x is zero. Note that this means that f has to be constant with respect to x, do the same thing for y and find a contradiction.

7. Prove that the quadratic function $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = x^2$ is continuous at $x = 1$. (You need the find a relationship between the ε and the δ).

[2] Google space filling curves if you want your mind blown, are they differentiable?