## HARD HOMEWORK #2 – MATH 1260
## FALL 2014

SOLUTION TO PROBLEM #4

**4.** Suppose $A \subseteq \mathbb{R}^n$ is an open set and that $\vec{f} : A \longrightarrow \mathbb{R}^n$ is continuously differentiable and injective. Further suppose that the Jacobian matrix of $f$ always has non-zero determinant (for any point $\vec{x} \in A$). Show that $f(A)$ is open. Give an example which shows that $f(A)$ is not necessarily open if the Jacobian matrix of $f$ has zero determinant.

**Solution:** Choose $f(a)$ a random point in $f(A)$, here $a \in A$. We need to show that there is a small ball around $f(a)$ contained in $f(A)$. Since $f$ has jacobian with nonzero determinant at $a \in A$, we know there is an open set $V \in A$ containing $a$ and another open set $W$ in the codomain $\mathbb{R}^n$ so that $f : V \longrightarrow W$ has an inverse. In particular, $f(V) = W$ (functions with inverses need to be both one-to-one and onto). Now, since $V \subseteq A$, $W = f(V) \subseteq f(A)$. Since $W$ is open and contains $f(a)$ (since $a \in V$), we know there exists a small open ball around $f(a)$ in $W$, and so that small open ball is also in $f(A)$. But then $f(A)$ is open.

You could also reasonably ask for the example. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ which sends $f(\langle x, y\rangle)$ to $\langle 0, 0\rangle$. Then for any open nonempty set $U$, $f(U) = \{\langle 0, 0\rangle\}$ which is definitely not open.