HARD HOMEWORK #1 – MATH 1260
FALL 2014

DUE, MONDAY NOVEMBER 24TH

This homework is on the more theoretical aspects of Calculus that we have been discussing. Of course, I encourage you to work in groups, but each person needs their own write-up.

First let us recall some definitions.

Definition (Open) A set $U \subseteq \mathbb{R}^n$ is *open* if for every point $\vec{z} \in U$, there is a little open box[1] around z which contains U.

Definition (Closed) A set $U \subseteq \mathbb{R}^n$ is closed if its complement $\mathbb{R}^n \setminus U$ is open.

Definition (Open Cover) A (possibly infinite) collection of sets $\{U_i\}$, with each $U_i \subseteq \mathbb{R}^n$ an open set, is called an *open cover of* $A \subseteq \mathbb{R}^n$ if every point $\vec{z} \in A$ is in at least one U_i.

Definition (Compact) A subset $A \subseteq \mathbb{R}^n$ is called *compact* if every open cover has a finite subcover. We said (but didn't prove) that A is also continuous if and only if it is closed and bounded[2]

Definition (Continuous) A function $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called *continuous at* $\vec{z} \in \mathbb{R}^n$ if for every ball $B \subseteq \mathbb{R}^m$ of radius $\varepsilon > 0$ around $f(z)$, there is a ball $C \subseteq \mathbb{R}^n$ of radius $\delta > 0$ around z with $f(C) \subseteq B$.

Another way to say this is: For every $\varepsilon > 0$, there is a $\delta > 0$ so that if $|\vec{z} - \vec{z}| < \delta$ then $|f(\vec{z}) - f(\vec{x})| < \varepsilon$.

1. Suppose that $\{U_i\}$ is an infinite collection of open sets. Prove that the union $\bigcup_i U_i$ is also open.

2. Find an infinite collection of open sets $\{U_i\}$ such that $\bigcap_i U_i$ is *not* open.

3. Show that if $A \subseteq \mathbb{R}^n$ is not bounded, then A is *not compact*.

4. Prove that linear functions $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = mx + b$ are indeed continuous.

5. Suppose that $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ is a continuous and that $A \subseteq \mathbb{R}^m$ is compact. Show that $f(A)$ takes on a maximum value. This means that there is an $L \in f(A)$ with $L \geq x$ for all other $x \in f(A)$.

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[1] You call replace box with *ball*, think about why.

[2] Meaning that there is a $L > 0$ so that if $\vec{z} \in A$ then $|\vec{z}| < L$.