EXTRA CREDIT #3– MATH 1260
FALL 2014

DUE 11/21/2014

To receive credit on this assignment, your solutions needed to be *neatly* written up or typed.

Let $W \subseteq \mathbb{R}^n$ be a region. For instance, it could be the interior of the sphere in $\mathbb{R}^3$, it could be the surface of the sphere in $\mathbb{R}^3$, or even the surface of the doughnut in $\mathbb{R}^3$.

A *loop* C in D is a continuous function $\vec{r} : [0,1] \longrightarrow D$ with $\vec{r}(0) = \vec{r}(1)$. Note that a loop is oriented, meaning it has a direction. Here $[0,1]$ is the closed interval.

1. Explain in words why we need the condition $\vec{r}(0) = \vec{r}(1)$. (1 point)

2. Which of the following functions are loops and which are not? You need to explain your reasoning in order to receive credit. (2 points)

(a) $\vec{r} : [0,1] \longrightarrow \mathbb{R}^2$ defined by $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle$.
(b) $\vec{r} : [0,1] \longrightarrow \mathbb{R}^2$ defined by $\vec{r}(t) = \langle \cos(2\pi t), \sin(4\pi t) \rangle$.
(c) $\vec{r} : [0,1] \longrightarrow \mathbb{R}^2$ which sends $\vec{r}(t) = \langle \frac{1}{(t-0.5)^2}, \rangle$.
(d) $\vec{r} : [0,1] \longrightarrow \mathbb{R}^3$ which sends $\vec{r}(t) = \langle 4(t-0.5)^2, \cos(2\pi t), t(1-t) \rangle$.

3. Suppose that C and D are loops in $W \subseteq \mathbb{R}^n$ parameterized by $\vec{u}(t)$ and $\vec{v}(t)$. Further suppose the C and D have the same start-end point Q.

(a) We define $-C$ to be the loop obtained by going along C backwards. Write down a parameterization of $-C$ based on $\vec{u}(t)$ the parameterizaiton of $\vec{u}$. (1 point)
Hint: You need to find a function $g(t)$ so that $\vec{u}(g(t))$ parameterizes $-C$.
(b) We define the curve CD to be the (double) loop that first follows C and then follows D around. Write down a parameterization of CD. (1 point)
Hint: You can write a multipart function $\vec{r}(t) = \begin{cases} \dots & 0 \leq t \leq \frac{1}{2} \\ \dots & \frac{1}{2} \leq t \leq 1 \end{cases}$

We use $[0,1] \times [0,1]$ to denote the square $\{(s,t) | 0 \leq s \leq 1, 0 \leq t \leq 1\}$. Suppose that two loops C and D in $W \subseteq \mathbb{R}^n$ have the same start-end point Q. We say that C and D are *homotopic* if there is a continuous function $H : [0,1] \times [0,1] \longrightarrow W$ so that $H(0,t)$ parameterizes C and $H(1,t)$ parameterizes D. We also assume $H(s,0) = Q = \vec{h}(s,1)$.

4. Explain why saying that C and D are homotopic means we can *deform* C to D (think about plugging different s values in, what do you get?). Make sure to explain what the condition $H(s,0) = Q = H(s,1)$ means. (1 point)

5. Consider the following function $H(s,t) = \langle (1-s)(\cos(2\pi t) - 1) + s(2\cos(2\pi t) - 2), \sin(2\pi t) \rangle$. This is a homotopy between two curves. Describe the two curves that are demonstrated to be homotopic. (1 point)

6. Consider the circle of radius 1 as a loop counter-clockwise and the square parameterized so that it goes from the end points $(1,0) \longrightarrow (1,1) \longrightarrow (-1,1) \longrightarrow (-1,-1) \longrightarrow (1,-1) \longrightarrow (1,0)$. Draw a picture to show that these two loops are homotopic. (1 point)

Find a function H that really shows that these two loops are homotopic. (4 points)

7. Fix a start-end point $Q \in W$. We say that W is *simply connected* if every path is homotopic to every other path. Draw some pictures to show that $\mathbb{R}^2$ is simply connected. Is the surface of the sphere simply connected, draw some pictures to justify your answer? What about the annulus $\{(x,y) \mid 1 < x^2 + y^2 < 2\}$, again draw some pictures to justify your answer. (3 points).

8. Fix W and a start-end point $Q \in W$. Suppose that C_1 and C_2 are homotopic. Show that $C_1 D$ and $C_2 D$ are also homotopic, both by picture and with an explicit formula (2 points).

9. Given a loop C in W (with fixed base start-end point Q), we declare the homotopy equivalence class $[C]$ to be the set of all looops C' homotopic to C. Show that if $[C]$ and $[D]$ have any loops in common, then $[C] = [D]$ (2 points).

10. Describe four distinct homotopy equivalence classes in the annulus $\{(x,y) \mid 1 < x^2 + y^2 < 2\}$. (2 points)

11. Fix W and a start-end point $Q \in W$. We define the *fundamental group of* W to be the set of all homotopy equivalence classes with the multiplication operation $\star$ defined by

$$[C] \star [D] = [CD].$$

Our work in **8.** shows that this is well defined (the multiplication doesn't depend on the choices of representatives of the class). Describe all the distinct homotopy equivalence classes in the annulus $\{(x,y) \mid 1 < x^2 + y^2 < 2\}$, also describe how the $\star$ operation combines them. Is it true that $[C] \star [D] = [D] \star [C]$? Given examples, draw pictures, etc. (3 points)

12. Finally consider the surface of the doughnut and consider its fundamental group. Describe its fundamental group to the best of your ability. Answer the following question on that surface, is it true that $[C] \star [D] = [D] \star [C]$? (3 points)

Hint: This is hard, to visualize it, think about the surface of the doughnut parameterized like we did it in the first computer assignment. What do various loops look like in that parameterization?