INFO ON DETERMINANTS – MATH 1260
FALL 2014

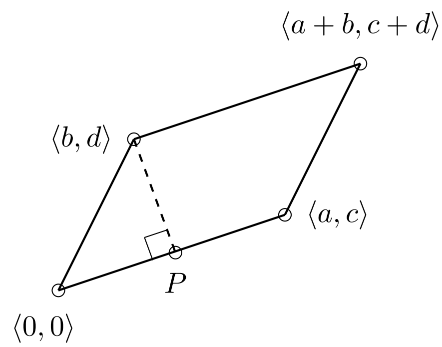
FROM CLASS, 8/29/14

We start with a function $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ (you can think of it as a function which takes vectors to other vectors).

Suppose it sends $\langle x, y\rangle$ to $\langle ax + by, cx + dy\rangle$. Such a function is called *linear* for fairly obvious reasons. It can be represented by a matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

This function takes a unit square (the region bounded by $x \geq 0$, $y \geq 0$, $x \leq 1$, $y \leq 1$) and transforms it to a parallelogram with corner points $\langle 0, 0\rangle, \langle a, c\rangle, \langle b, d\rangle, \langle a + b, c + d\rangle$.



Let us find the area of this parallelogram (the formula is bh). We see that the bottom side has length $\sqrt{a^2 + b^2}$ and so we only need to find the length of the dotted line. We can project $\langle b, d\rangle$ onto $\langle a, c\rangle$ to find the length of $\vec{OP}$. That is just

$$\frac{\langle a, c\rangle \cdot \langle b, d\rangle}{|\langle a, c\rangle|} = \frac{ab + cd}{\sqrt{a^2 + c^2}}.$$

By the pythagorean theorem, the dotted line then has length

$$\sqrt{\left(\sqrt{b^2 + d^2}\right)^2 - \left(\frac{ab + cd}{\sqrt{a^2 + c^2}}\right)^2} = \sqrt{\frac{(b^2 + d^2)(a^2 + c^2) - (ab + cd)^2}{a^2 + c^2}} = \sqrt{\frac{b^2c^2 + a^2d^2 - 2abcd}{a^2 + c^2}}$$

which is simply

$$\sqrt{\frac{(ad - bc)^2}{a^2 + c^2}}.$$

Therefore, the area of the parallelogram is simply

$$\left(\sqrt{a^2 + c^2}\right) \cdot \left(\sqrt{\frac{(ad - bc)^2}{a^2 + c^2}}\right) = |ad - bc|.$$

Indeed, if we just use the term $ad - bc$, then this measures both the area of the parallelogram and whether or not the parallelogram has the reverse orientation compared to the square (if $\langle a, c\rangle$ is the upper left corner of the parallelogram even if it is the lower right corner of the square).

Definition 0.1. The determinant of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, or of the function T which the matrix represents, is defined to be $ad - bc$.

Likewise suppose you are given a linear function $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which sends

$$\langle x, y, z \rangle \mapsto \langle ax + by + cz, dx + ey + fz, gx + hy + iz \rangle$$

and so is represented by a matrix

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

then to determine the volume T(unit cube) you compute the determinant of A, denoted $\det(A)$. We won't derive the formula, I'll just give you the rule.

Consider

$$\begin{bmatrix} \mathbf{a} & & \\ & e & f \\ & h & i \end{bmatrix}, \begin{bmatrix} & \mathbf{b} & \\ d & & f \\ g & & i \end{bmatrix}, \begin{bmatrix} & & \mathbf{c} \\ d & e & \\ g & h & \end{bmatrix}$$

and compute the determinants of the 2×2 matrices embedded above. In other words, compute

$$\det \begin{bmatrix} e & f \\ h & i \end{bmatrix}, \det \begin{bmatrix} d & f \\ g & i \end{bmatrix}, \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}.$$

Next multiply them by a, b, c and add them up with alternate signs.

$$\det A = a \cdot \det \begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \cdot \det \begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \cdot \det \begin{bmatrix} d & e \\ g & h \end{bmatrix}.$$

That gives you

$$\det A = a(ei - fh) - b(di - fg) + c(dh - eg).$$

I *do not* expect you to memorize such a formula. But you should be able to derive it. The point is that that number again tells you how much volume changes under T.

If we have even higher dimensions, the same basic idea still applies.

Given

$$B = \begin{bmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{bmatrix}$$

corresponding to a map $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$, we have that

$$\det(B) = a \cdot \det \begin{bmatrix} f & g & h \\ j & k & l \\ n & o & p \end{bmatrix} - b \cdot \det \begin{bmatrix} e & g & h \\ i & k & l \\ m & o & p \end{bmatrix} + c \cdot \det \begin{bmatrix} e & f & h \\ i & j & l \\ m & n & p \end{bmatrix} - d \cdot \det \begin{bmatrix} e & f & g \\ i & j & k \\ m & n & o \end{bmatrix}.$$

And so one can compute how much "4-dimensional volume" changes under T.