F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/9-2010

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1. Frobenius splittings of projective varieties and graded rings

Given a projective variety X with a ((very) ample) Cartier divisor A, we can construct the section ring

$$S := \oplus_{n\geq 0}\mathcal{O}_X(nA).$$

Likewise, given an $\mathcal{O}_X$-module $\mathscr{M}$, we can construct $M := \oplus_{n\geq 0}\mathscr{M}(n)$ where $\widetilde{M} = \mathscr{M}$ (see for example [Har77]).

If $\mathscr{L}$ is a very ample divisor corresponding to an embedding into $\mathbb{P}^n$ with associated section ring S, then S may or may not agree with the affine cone of X (in $\mathbb{P}^n$). If X is normal, then the affine cone and $\operatorname{Spec} S$ agree if and only if the embedding is projectively normal. However, if $\mathscr{L}$ is sufficiently ample, then the two rings agree.

If X is an F-finite scheme, we can consider $F^e_*\mathcal{O}_X$ and the associated module $M := \oplus_{n\geq 0}(F^e_*\mathcal{O}_X)(n)$ and compare it with F^e_*S.

Question 1.1. Is M isomorphic to F^e_*S as a graded S-module?

We'll answer this question with an example.

Example 1.2. Consider $X = \mathbb{P}^1_k$, $k = \bar{k}$ with the usual ample divisor $\mathcal{O}_X(1)$. In this case, $M = \oplus_{n\geq 0}F^e_*\mathcal{O}_X(np^e)$ which is quite different from $F^e_*S = \oplus_{n\geq 0}F^e_*\mathcal{O}_X(n)$ (in F^e_*S, some graded pieces are k-vector spaces of dimension p).

One should note that F^e_*M is not a $\mathbb{Z}$-graded S-module. It is instead a $\mathbb{Z}[1/p^e]$-graded S-module. By $[F^e_*M]_{n=0 \bmod \mathbb{Z}}$ we mean the direct summand of F^e_*M with integral coefficients.

With this in mind.

Lemma 1.3. *Given a saturated S-module M corresponding to a coherent sheaf $\mathscr{M}$, we have an isomorphism of S-modules $[F^e_*M]_{n=0 \bmod \mathbb{Z}} \cong \oplus_{n\geq 0}(F^e_*\mathscr{M})(n)$.*

This yields the following interesting result.

Proposition 1.4. *Suppose that X is an F-finite F-split scheme, and $\mathscr{L}$ is any line bundle. Then the section ring*

$$S := \oplus_{i\geq 0}H^0(X, \mathscr{L}^i)$$

is also Frobenius split.

Proof. We have the following splittings for all $i \geq 0$

$$\mathscr{L}^i \to F_*\mathscr{L}^{ip^e} \to \mathscr{L}^i$$

where the composition is an isomorphism and the first map is Frobenius. This implies that $S \to [F_*S]_{n=0 \bmod \mathbb{Z}}$ splits. But $[F_*S]_{n=0 \bmod \mathbb{Z}} \to F_*S$ also clearly splits. Composing these splittings gives the desired result. □

The converse to the previous proposition also holds if $\mathscr{L}$ is ample.

Theorem 1.5. *Suppose that X is an F-finite F-split scheme, $\mathscr{L}$ is an ample line bundle, and S is the section ring of X with respect to $\mathscr{L}$. If S is Frobenius split, then so is X.*

We will prove this in stages. The first stage allows us to assume that $\mathscr{L}$ is (very very) ample (which isn't strictly necessary, but it is harmless and easy regardless).

Lemma 1.6. *If S is a Frobenius split graded ring, then any veronese subring is also Frobenius split.*

Proof. Suppose that $S_{(n)}$ is the nth veronese subring of S. The map $S_{(n)} \subseteq S$ is clearly split, thus $S_{(n)}$ is Frobenius split as well. □

Remark 1.7. If $S_{(n)}$ is Frobenius split and p does not divide n, then S is also split as we will see later (the Veronese map is étale in codimension 1 in this case).

Lemma 1.8. *If S is a Frobenius split graded ring, then S has a "graded" Frobenius splitting.*

Proof. To define a graded Frobenius splitting, we first have to remind ourselves what the grading on F_*S is. Remember, F_*S is $\mathbb{Z}[1/p]$-graded, which makes the Frobenius map $S \to F_*S$ a degree preserving graded map. A graded splitting is thus going to be a graded (degree preserving) map $F_*S \to S$ that sends 1 to 1. Since S is split, there are obviously plenty of (possibly non-graded) maps $\phi : F_*S \to S$ which sends 1 to 1. We simply have to find a graded such map.

On the other hand, we have the *evaluation-at-1* map $\mathrm{Hom}_S(F_*S, S) \to S$. Because S is F-finite, the module $\mathbb{Z}[1/p]$-graded $\mathrm{Hom}_S(F_*S, S)$ is generated over S_0 by graded but degree shifting maps $F_*S \to S$. So suppose ϕ is an arbitrary splitting. We can write $\phi = \phi_0 + \cdots + \phi_n$ where ϕ_n are degree shifting maps and ϕ_0 is degree preserving (this is a basic commutative algebra fact, a proof can be found in [BH93, Section 1.5]). It is clear that $\phi_0(1) = 1$ because $\phi(1)$ equals 1 and none of the other $\phi_i(1)$ can possibly land in the correct degree. Thus ϕ_0 is our desired degree preserving splitting. □

Proof of Theorem 1.5. We may assume that $\mathscr{L}$ is very (very) ample and so our ring standard graded (generated in degree 1). We have the following composition

$$S \to [F_*S]_{n=0 \mod \mathbb{Z}} \to F_*S$$

By the previous lemma, this composition has a degree preserving graded splitting. Thus $S \to [F_*S]_{n=0 \mod \mathbb{Z}}$ also has a degree preserving graded splitting. Thus $\mathcal{O}_X = \widetilde{S} \to \widetilde{F_*S} = F_*\mathcal{O}_X$ also splits (as desired). □

Finally, let us give an example to elliptic curves. We have already seen that a supersingular elliptic curve cannot be F-split (ie, an F-split elliptic curve must be ordinary), we will now prove the converse.

First we recall that $\mathscr{H}om_X(F^e_*\mathcal{O}_X, \omega_X) \cong \omega_X$ (as we did this before, this was non-canonical) for X a variety over an algebraically closed (or even F-finite) field. Thus, applying the functor $\mathscr{H}om_X(\_, \mathcal{O}_X)$ to $F : \mathcal{O}_X \to F_*\mathcal{O}_X$ gives us a map $F^e_*\omega_X \to \omega_X$ (sometimes called the trace map).

Proposition 1.9. *Suppose that X is an ordinary [1]elliptic curve, then X is F-split.*

[1]Ordinary means that $F : H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X)$ is injective.

Proof. We know that $H^1(X, \mathcal{O}_X) \to H^1(X, F_*\mathcal{O}_X)$ is injective. Serre-duality tells us that $H^0(X, F_*\omega_X) \to H^0(X, \omega_X)$ is surjective (where this map is induced by what we called the trace map above, one can see this via Grothendieck duality or by a degenerating spectral sequence argument). But on an elliptic curve, $\omega_X \cong \mathcal{O}_X$ so that we have a map $\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$ that is surjective on global sections. In particular, there is a global section of $F_*\mathcal{O}_X$ which is sent to 1 by ϕ. This element is just a unit, and so by rescaling, we can assume that ϕ sends 1 to 1 and is thus a splitting. This means □

We also do an example of these ideas for $\mathbb{P}^n$.

Example 1.10. Suppose that $X = \mathbb{P}^n_k$ where $k = \bar{k}$. For $n = 1$ we already computed $F_*\mathcal{O}_X$. Let us at least show that $F^e_*\mathcal{O}_X$ is a direct sum of line bundles for $n > 1$ (this is an old result due to Hartshorne). Let S denote the section ring with respect to the usual $\mathcal{O}(1)$ (so that $S = k[x_0, \ldots, x_n]$. We have the graded module $M := \oplus_{n \geq 0}(F^e_*\mathcal{O}_X)(n)$ which we know is a summand of $F^e_*\mathcal{O}_S$/ However, $F^e_*\mathcal{O}_S$ is a free S-module, which implies that M is projective and thus also a free S-module because M is graded (see for example [BH93, Proposition 1.5.15(d)]). So write $M = \oplus S(i)$ for various i. Therefore $F^e_*\mathcal{O}_X = \widetilde{M} = \oplus \widetilde{S(i)} = \oplus \mathcal{O}_X(i)$.

We also give an example related to projective normality. Recall that on the first day of class we showed that if $Z \subseteq \mathbb{P}^n$ is compatibly Frobenius split in $X = \mathbb{P}^n$, then it's embedding is projectively normal (meaning in this case that $H^0(X, \mathcal{O}_X(i)) \to H^0(Z, \mathcal{O}_Z(i))$ is surjective for all i, this always happens for a good enough veronese). We will prove a partial converse to this statement.

Proposition 1.11. *Suppose that Z is a Frobenius split variety embedded (projectively normally) in $X = \mathbb{P}^n$. Then Z is compatibly Frobenius split in $X = \mathbb{P}^n$.*

Proof. In fact, we will show that any Frobenius splitting of Z extends to one on $\mathbb{P}^n$. Fix $\phi : F^e_*\mathcal{O}_Z \to \mathcal{O}_Z$ to be a map. This induces a graded degree-preserving map $\Phi : \oplus H^0(X, (F^e_*\mathcal{O}_Z)(i)) \to R$ on the section ring $R = \oplus H^0(X, \mathcal{O}_Z(i))$ as we've seen. However, because of the projective normality assumption, R is a quotient of $S = \oplus H^0(X, \mathcal{O}_X(i))$ (this means that the affine cone and the section ring coincide). But S is a polynomial ring and so a graded version of the proof of Fedder's Lemma implies that Φ extends to a map $\bar{\Phi} : \oplus H^0(X, (F^e_*\mathcal{O}_X)(i)) \to S$ (and we may assume that this map is also graded and degree preserving). Using the $\tilde{\ }$ operation gives us our splitting on X which is compatible with the one on $\mathbb{Z}$. □

We'll later see that Frobenius splitting has some analog with regards to log Calabi-Yau varieties. Furthermore, I know of no analog of this statement in the log-Calabi-Yau context.

References

[BH93] W. Bruns and J. Herzog: *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. MR1251956 (95h:13020)

[Har77] R. Hartshorne: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)