F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/7-2010

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1. (Weak/Semi)normality and Frobenius splitting

Today we'll prove that a F-split ring is weakly normal and thus seminormal (so first I'll define these terms).

First we'll talk about some hand-wavy geometry. Seminormality (and weak normality) are ways of forcing all gluing of your scheme is as transverse as possible. So first what is "gluing"?

Suppose that R is an F-finite reduced ring with normalization R^N (domain of finite type over a field is fine). The semi-normalization R^{SN} (and weak normalization R^{WN} of R is a partial normalization of R inside R^N). Since R is F-finite it is excellent, and so all these extensions are finite extensions (ie, we don't have to worry about extreme funny-ness).

Definition 1.1. [AB69], [GT80], [Swa80] A finite integral extension of reduced rings $i : A \subset B$ is said to be *subintegral* (respectively *weakly subintegral*) if

(i) it induces a bijection on the prime spectra, and
(ii) for every prime $P \in \operatorname{Spec} B$, the induced map on the residue fields, $k(i^{-1}(P)) \to k(P)$, is an isomorphism (respectively, is a purely inseparable extension of fields).

Remark 1.2. A subintegral extension of rings has also been called a quasi-isomorphism; see for example [GT80].

Remark 1.3. Condition (ii) is unnecessary in the case of extensions of rings of finite type over an algebraically closed field of characteristic zero.

Definition 1.4. [GT80, 1.2], [Swa80, 2.2] Let $A \subset B$ be a finite extension of reduced rings. Define ${}^+_BA$ to be the (unique) largest subextension of A in B such that $A \subset {}^+_BA$ is subintegral. This is called the *seminormalization of A inside B*. A is said to be *seminormal in B* if $A = {}^+_BA$. If A is seminormal inside its normalization, then A is called *seminormal.*

Definition 1.5. [AB69], [Yan85], [RRS96, 1.1] Let $A \subset B$ be a finite extension of reduced rings. Define ${}^*_BA$ to be the (unique) largest subextension of A in B such that $A \subset {}^*_BA$ is weakly subintegral. This is called the *weak normalization of A inside B*. A is said to be *weakly normal in B* if $A = {}^*_BA$. If A is weakly normal inside its normalization, then A is called *weakly normal.*

Remark 1.6. Note the following set of implications.

$$\text{Normal} \Longrightarrow \text{Weakly Normal} \Longrightarrow \text{Seminormal}$$

Consider the following examples:

(i) The union of two axes in $\mathbb{A}^2$, $\operatorname{Spec} k[x, y]/(xy)$, is both weakly normal and seminormal, but not normal (an irreducible node is seminormal as well).

(ii) The union of three lines through the origin in $\mathbb{A}^2$, $\operatorname{Spec} k[x, y]/(xy(x - y))$, is neither seminormal nor weakly normal.

(iii) The union of three axes in $\mathbb{A}^3$, $\operatorname{Spec} k[x, y, z]/(x, y) \cap (y, z) \cap (x, z)$, is both seminormal and weakly normal. In fact, it is isomorphic to the seminormalization of (ii).

(iv) The pinch point $\operatorname{Spec} k[a, b, c]/(a^2 b - c^2) \cong \operatorname{Spec} k[x^2, y, xy]$ is both seminormal and weakly normal as long as the characteristic of k is not equal to two. In the case that $\operatorname{char} k = 2$, then the pinch point is seminormal but not weakly normal. Notice that if $\operatorname{char} k = 2$ then the inclusion $k[x^2, y, xy] \subset k[x, y]$ induces a bijection on spectra. Furthermore the induced maps on residue fields are isomorphisms at all closed points. However, at the generic point of the singular locus $P = (y, xy)$, the induced extension of residue fields is purely inseparable. This proves that it is not weakly normal.

(v) $\mathbb{R}[x, y]/(x^2 + y^2)$ is seminormal and weakly normal (even though the residue field changed).

A useful way to construct examples is the following lemma.

Lemma 1.7. *Suppose that A is a ring, $I \subseteq$ is an ideal and B is another ring with a map $\phi : B \to A/I$. Then the pullback C of the diagram of rings*

$$\{A \to A/I \leftarrow B\}$$

has the following properties.

(i) $\operatorname{Spec} C$ *has a closed subscheme W that maps isomorphically to* $\operatorname{Spec} B$ *via the induced map (from the pullback diagram).*

(ii) *The induced map $C \to A$ gives an isomorphism between* $(\operatorname{Spec} C) \setminus W$ *and* $(\operatorname{Spec} A) \setminus (\operatorname{Spec} A/I)$.

(iii) *As topological spaces,* $\operatorname{Spec} C$ *is the pushout of the (dual) diagram of* Spec*'s.*

All of the examples from the previous remark can be constructed as pull-backs in this way.

There are other characterizations of weakly normal and seminormal which are of a more algebraic nature, and are often very useful. We'll only prove the second one.

Proposition 1.8. [LV81, 1.4] *Let $A \subset B$ be a finite integral extension of reduced rings; the following are then equivalent:*

(i) *A is seminormal in B*

(ii) *For a fixed pair of relatively prime integers $e > f > 1$, A contains each element $b \in B$ such that $b^e, b^f \in A$. (also see* [Ham75] *and* [Swa80] *for the case where $e = 2$, $f = 3$).*

Proposition 1.9. [RRS96, 4.3, 6.8] *Let $A \subset B$ be a finite integral extension of reduced rings where A contains $\mathbb{F}_p$ for some prime p; the following are then equivalent:*

(i) *A is weakly normal in B.*

(ii) *If $b \in B$ and $b^p \in A$ then $b \in A$.*

Proof. First we show that (i) implies (ii). Suppose, for a contradiction, that there is a $b \in B$ such that $b^p \in A$ but $b \notin A$. We will show that $A[b]$ is subintegral over A. Observe that for any element $f \in A[b]$, we know that $f^p \in A$.

First suppose that $P \in \operatorname{Spec} A$, we will show that there is exactly one prime $Q \in \operatorname{Spec} A[b]$ lying over P (obviously there is at least one and at most finitely many). Suppose that $cd \in \sqrt{P \cdot \operatorname{Spec} A[b]}$, then $(cd)^n \in P \cdot \operatorname{Spec} A[b]$ and so even better, $(cd)^{pn} \in (P \cdot \operatorname{Spec} A[b]) \cap A = P$, thus $c^p \in P$ or $d^p \in P$ by the primality of P. But if $c^p \in P$ then $c \in \sqrt{P \cdot \operatorname{Spec} A[b]}$ and likewise with d. This proves that at least the spec's line up. The residue field extensions are even easier since $A_P/P \subseteq (A[b]_{\sqrt{P \cdot A[b]_P}}/\sqrt{P \cdot A[b]_P})$ is obviously a field extension generated by a purely inseparable element (if the extension is non-trivial).

Conversely, suppose that $A \subseteq B$ is not a weakly normal extension. Thus we may assume that it is a weakly subintegral extension. Choose $b \in B$ such that $b \notin A$. It is sufficient to show that $b^{p^e} \in A$ for some $e > 0$. But first we make several reductions. Note that if condition (ii) holds on $A \subseteq B$, then it also holds after localizing at a multiplicative subset. To see this, note that if $b \in B$, $(b/s) \in S^{-1}B$ and $(b/s)^p \in S^{-1}A$, then by assumption $s^n(b/s)^p \in A$ for some n (we may assume $n = pm$). Thus $(s^{m-1}b)^p \in A$ and $s^{m-1}b \in B$ so that $s^{m-1}b \in A$ by assumption. Thus $b/s \in A$ also. Consider the locus of $\operatorname{Spec} A$ over which A is not weakly normal in B (this locus is closed – it's just the conductor of $A \subseteq B$), by localizing, we may assume that this is the maximal ideal of the local ring A. Thus $A \subseteq B$ induces a bijection on points of Spec and, all residue field extensions are trivial or purely inseparable.

Furthermore, since the extension is already both weakly subintegral and also weakly normal except at the maximal ideal, it is an isomorphism except at the maximal ideal. It follows that the residue field extension at the maximal ideal is purely inseparable. Now, consider the pull-back C of the following diagram.

$$\{B \to B/\mathfrak{m}_B \leftarrow A/\mathfrak{m}_A\}$$

This pullback C agrees with A except at the origin possibly (and by the universal property of pull-backs, we have $A \subseteq C$). However, by (ii), the extension is seminormal and since $A \subseteq C$ is clearly subintegral, must be an isomorphism. Choose $b \in B$, then $\bar{b}^{p^e} \in A/\mathfrak{m}_A$ for some $e > 0$, thus $b^{p^e} \in C$ for that same $e > 0$. □

Theorem 1.10. [HR76] *If R is F-split, then it is weakly normal.*

Proof. Suppose that $r \in R^N$ and $r^p \in R$. We have the splitting $\phi : F^e_* R \to R$ which sends 1 to 1. Thus $\phi(r^p) = r\phi(1) = r$ so that $r \in R$ as well. □

We now prove a partial converse in the one-dimensional case. A special case of this can be found in [GW77]. First we need a lemma.

Lemma 1.11. *If $K \subseteq L$ is a finite separable extension of fields, then any map $\phi : F^e_* K \to K$ uniquely extends to a map $\bar{\phi} : F^e_* L \to L$.*

Proof. Left to the exercises. □

Remark 1.12. In fact if $K \subseteq L$ is not separable, then the only map $F^e_* K \to K$ which extends to a map $F^e_* L \to L$ is the zero map.

Theorem 1.13. *If R one dimensional, F-finite and weakly normal with a perfect residue field, then it is F-split.*

Proof. In this proof, we will effectively classify one dimensional F-split varieties with perfect residue fields. It is harmless to assume that R is local with maximal ideal $\mathfrak{m}$ and residue field k. Let R^N denote the normalization of R. We may write $R^N = R_1 \oplus \cdots \oplus R_m$ where each R_i is a semi-local ring with maximal ideals $\mathfrak{m}_{i,1}, \ldots, \mathfrak{m}_{i,n_i}$ and residue fields $k_{i,1}, \ldots, k_{i,n_i}$ (each of which is a finite, and thus separable, extension of k).

We also have the pullback diagram

$$\{R^N = R_1 \oplus \cdots \oplus R_m \to (R_1/\mathfrak{m}_1) \oplus \cdots \oplus (R_m/\mathfrak{m}_m) = k_{1,1} \oplus \cdots \oplus k_{m,n_m} \leftarrow k\}$$

The pullback C of this diagram is an extension ring of R. It is also clearly a subintegral extension of R so $R = C$. Thus we will show that C is F-split. Choose a map $\phi : F^e_* k \to k$ that is non-zero. On each k_{i,n_i}, this map extends to a map $\phi_{i,n_i} : F^e_* k_{i,n_i} \to k_{i,n_i}$. Because each R_i is a semi-local regular ring, by Fedder's Lemma, each $\phi_{i,n_i} : F^e_* R_i/(\cap_t \mathfrak{m}_{i,t}) \to R_i/(\cap_t \mathfrak{m}_{i,t})$ extends to a map $\psi_{i,n_i} : F^e_* R_i \to R_i$. These maps then "glue" to a map on C. □

Based on the previous result, it is natural to ask whether every $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ extends to a map on the normalization? We will show that the answer is yes, but first we show a result about the conductor.

Proposition 1.14. [BK05, Exercise 1.2.E] *Given a reduced F-finite ring R with normalization R^N, the conductor ideal of R in R^N is ϕ-compatible for every $\phi \in \operatorname{Hom}_R(F^e_* R, R)$.*

Proof. The conductor ideal I can be defined as "the largest ideal $I \subseteq R$ that is simultaneously an ideal of R^N". It can also be described as

$$I := \operatorname{Ann}_R R^N/R = \{x \in R | xR^N \subseteq R\}.$$

Following the proof of [BK05, Proposition 1.2.5], consider $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. Notice, that by localization, ϕ extends to a map on the total field of fractions (which contains R^N). We will abuse notation and also call this map ϕ (since it restricts to $\phi : F^e_* R \to R$). Now choose $x \in F^e_* I$ and $r \in R^N$. Then $\phi(x)r = \phi(xr^{p^e}) \in \phi(F^e_* R) \subseteq R$. Thus $\phi(x) \in I$ as desired. □

Proposition 1.15. [BK05, Exercise 1.2.E(4)] *For a reduced F-finite ring R, every map $\phi : F^e_* R \to R$, when viewed as a map on total field of fractions, restricts to a map $\phi' : F^e_* R^N \to R^N$ on the normalization.*

Proof. We follow the hint for [BK05, Exercise 1.2.E(4)]. For any $x \in R^N \in K(R)$, we wish to show that $\phi(x) \in R^N$. First we show that we can reduce to the case of a domain. We write $R \subseteq K(R) = K_1 \oplus \cdots \oplus K_t$ as a subring of its total field of fractions. Since each minimal prime Q_i of R is ϕ-compatible, it follows that ϕ induces a map $\phi_i : F^e_* R/Q_i \to R/Q_i$ for each i. Notice that the normalization of $\operatorname{Spec} R$ is a disjoint union of components (each one corresponding to a minimal prime of R), and the ith component is equal to $\operatorname{Spec}(R/Q_i)^N$. Thus, since we are ultimately interested in ϕ' restricted to each $(R/Q_i)^N$, it is harmless to assume that R is a domain.

Suppose that I is the conductor and consider $I\phi(x)$. For any $z \in I$, $z\phi(x) = \phi(z^{p^e}x) \in \phi(F^e_* I) \subseteq I$ (notice that $z^{p^e}x \in I$ since I is an ideal of R^N). More generally, $z\phi(x)^m = z\phi(x)(\phi(x))^{m-1} \subseteq I(\phi(x))^{m-1}$ which implies that $I\phi(x)^m \subseteq I\phi(x)^{m-1}$, and so by induction $I\phi(x)^m \subseteq I \subset R$ for all $m > 0$. This implies that for every $c \in I \subseteq R$ we have that $c\phi(x)^m \in I \subseteq R$. Therefore $\phi(x)$ is integral over R by [HS06, Exercise 2.26(iv)]. □

In particular, if R is F-split, then its normalization is also F-split. This proof (and the way we showed it) imply the following more general result.

Theorem 1.16. *An F-finite weakly normal one-dimensional local ring is F-split if and only if every residue field extension of $R \subseteq R^N$ is separable.*

Proof. Suppose that $(R, \mathfrak{m})$ is the local ring in question. If every residue field extension of R in R^N is separable, then the proof of Theorem 1.13 implies that R is F-split.

Conversely, if R is F-split, then there exists a surjective map $\phi : F_* R \to R$ which extends to a map $\bar{\phi} : F_* R^N \to R^N$ and which is compatible with I, the conductor ideal of $R \subseteq R^N$ (note that the induced map on R/I is non-zero). Since R is local and weakly normal, I is a radical ideal and thus $I = \mathfrak{m}$. Furthermore, I is a radical ideal on R^N and so it is a finite intersection of maximal ideals. In particular, the map ϕ restricted to $R/I = R/\mathfrak{m} = k$ extends to a map on the direct sum of its residue field extensions $R^N/I = k_1 \oplus \cdots \oplus k_n$. In particular, it extends to each k_i. But we know the map $\phi/\mathfrak{m} : F_* k \to k$ is non-zero, and since it extends to a map $F_* k_i \to k_i$ it follows that each k_i is a separable extension of k. ☐

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