F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/21-2010

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1. Reduction to characteristic $p > 0$

Note that if one also has the coordinates of a point $x \in X$ (closed or not), one can reduce that closed point to characteristic p as well. Let $x \subset R$ be a prime ideal of R and simply include coefficients for a set of generators of x into A. This gives us an ideal $x_A \in R_A$. Note without loss of generality we may assume that $x_A = x \cap R_A$ so that x_A is prime. Furthermore, we may assume that if we tensor the short exact sequence

$$0 \to x_A \to R_A \to R_A/x_A \to 0$$

by $\otimes_A \mathbb{C}$ we simply re-obtain

$$0 \to x \to R \to R/x \to 0,$$

the original exact sequence. Note that we may certainly also assume that R_A/x_A is A-free as well. In particular, if x is maximal (closed) and if we are working over $\mathbb{C}$ or any other algebraically closed field of characteristic zero, we may assume that $R_A/x_A = A$ since $R/x \cong \mathbb{C}$. Otherwise (still in the case where x is maximal) we see that R_A/x_A is a module-finite extension of A.

The following lemma is very useful for reducing cohomology to prime characteristic, the method of proof is essentially the same as [Har77, Chapter III, Section 12] (just different modules are flat).

Lemma 1.1. [Har98, 4.1] *Let X be a noetherian separated scheme of finite type over a noetherian ring A, and let $\mathscr{F}$ be a quasi-coherent sheaf on X, flat over A. Suppose that $H^i(X, \mathscr{F})$ is a flat A-module for each $i > 0$. Then one has an isomorphism*

$$H^i(X, \mathscr{F}) \otimes_A k(t) \cong H^i(X_{k(t)}, \mathscr{F}_{k(t)})$$

for every point $t \in T = \operatorname{Spec} A$ and $i \geq 0$, where $k(t)$ is the residue field of $t \in T$, $X_{k(t)} = X \times_T \operatorname{Spec}(k(t))$, and $\mathscr{F}_{k(t)}$ is the induced sheaf on $X_{k(t)}$.

Remark 1.2. In particular, by the previous lemma and flat base change, we see that $H^i(X_{\mathbb{C}}, \mathscr{F}_{\mathbb{C}}) = 0$ if and only if $H^i(X_{k(t)}, \mathscr{F}_{k(t)}) = 0$ for an open set of $t \in \operatorname{Spec} A$.

By making various cokernels of maps free A-modules, we may also assume that maps that are surjective over $\mathbb{C}$ are still surjective over A, and thus surjective in our characteristic p model as well. The following example illustrates this.

Example 1.3. Suppose we are given a scheme X with a divisor $E \subseteq X$ all over $\mathbb{C}$. Suppose that some map

$$H^i(X, \mathcal{O}_X) \to H^i(E, \mathcal{O}_E)$$

surjects for some i. Then we consider the corresponding map

$$H^i(X_A, \mathcal{O}_{X_A}) \to H^i(E_A, \mathcal{O}_{E_A}) \to C \to 0$$

with cokernel C. We may of course localize so that C is locally free over A, in which case, since tensoring with $\mathbb{C}$ over A cannot annihilate a non-zero element, we obtain that $C = 0$. Therefore the corresponding map was surjective in the first place. Then, since tensor is right exact, we apply 1.1 and obtain that the map

$$H^i(X_t, \mathcal{O}_{X_t}) \to H^i(E_t, \mathcal{O}_{E_t})$$

surjects as well.

Definition 1.4. Given a class of singularities P defined in characteristic $p > 0$, we say that a variety X in characteristic 0 has singularities of *open P-type* if for all sufficiently large choices of A as above, and all but finitely many maximal ideal $\mathfrak{p} \in A$, $X_{\mathfrak{p}}$ has P-singularities. We say that X in characteristic zero has singularities of *desne P-type* if for all sufficiently large choices of A as above, there exists a Zariski-dense set of maximal ideals $\mathfrak{p} \in \operatorname{Spec} A$ such that $X_{\mathfrak{p}}$ has P-singularities. In this way we can define singularities of (open/dense) F-rational, F-injective and F-split/pure type.

Remark 1.5. In general, the singularities we consider are stable under base change by finite field extensions, so one only needs to check a single finitely generated $\mathbb{Z}$-algebra A.

Theorem 1.6. *Suppose that X is a variety of characteristic zero. Then if X has dense F-rational type, X has rational singularities.*

Proof. Take a resolution $\pi : \widetilde{X}t\emptyset X$. The map $\omega_{\widetilde{X}} \to \omega_X$ surjects if and only if it's reduction to characteristic $p \gg 0$ does (and we've already shown that). The Cohen-Macaulay condition was done in the example above. ☐

Let's do another example of this sort of proof. We give another definition.

Definition 1.7. Suppose that X is a normal Cohen-Macaulay variety of characteristic zero and suppose that $\pi : \widetilde{X} \to X$ is a log resolution, fix E to be the exceptional divisor. We say that X has *Du Bois singularities* if $\pi_*\omega_{\widetilde{X}}(E) = \omega_X$.

Remark 1.8. Du Bois singularities can be defined for even reduced varieties, but the definition (and proofs) are much harder.

Theorem 1.9. *Suppose that X is normal, Cohen-Macaulay and has dense F-injective type, then X has Du Bois singularities.*

Proof. Let $\pi : \widetilde{X} \to X$ be a log resolution of X with exceptional divisor E. We reduce this entire setup to characteristic $p \gg 0$ such that the corresponding X is F-injective. Let $F^e : X \to X$ be the e-iterated Frobenius map.

We have the following commutative diagram,

$$\begin{array}{ccc} F^e_*\pi_*\omega_{\widetilde{X}}(p^eE) & \longrightarrow & \pi_*\omega_{\widetilde{X}}(E) \\ \rho\big\downarrow & & \big\downarrow\beta \\ F^e_*\omega_X & \xrightarrow{\ \phi\ } & \omega_X \end{array}$$

where the horizontal rows are induced by the dual of Frobenius, $\mathcal{O}_X \to F^e_* \mathcal{O}_X$ and the vertical arrows are the canonical maps. By hypothesis, ϕ is surjective. On the other hand, for $e > 0$ sufficiently large, the map labeled ρ is an isomorphism. Therefore the map $\phi \circ \rho$ is surjective which implies that the map β is also surjective. But then it must have been surjective in characteristic zero as well, and in particular, X has Du Bois singularities. $\square$

Remark 1.10. The above theorem also holds without the Cohen-Macaulay and normal hypotheses, but the proof is much more difficult.

2. Rational singularities are open F-rational type

Our main goal will to be to give a proof (modulo a hard theorem) of the following.

Theorem 2.1. [Har98], [MS97] *If X is in characteristic zero has rational singularities, then it has open F-rational type.*

To prove this, we will use the following lemma which we will black-box for today. First recall that on a normal variety X, a $\mathbb{Q}$-divisor is just an element of $\operatorname{div}(X) \otimes \mathbb{Q}$, a formal sum of prime divisors.

Lemma 2.2. [Har98] *Suppose that R_0 is a ring of characteristic zero, $\pi : X_0 \to \operatorname{Spec} R_0$ is a log resolution of singularities, D_0 is a π-ample $\mathbb{Q}$-divisor with simple normal crossings support. We reduce this setup to characteristic $p \gg 0$. Then the natural map*

$$(F^e)^\vee = \Phi_{X_p} : F^e_* \omega_{X_p}(\lceil p^e D_p \rceil) \to \omega_{X_p}(\lceil D_p \rceil)$$

surjects.

Before we use this, let us explain some points. We will assume that $\pi : X_0 \to \operatorname{Spec} R_0$ is projective, and thus the blow-up of some ideal sheaf $J \subseteq R$. It follows then that $J \cdot \mathcal{O}_{X_0} = \mathcal{O}_{X_0}(-F)$ is relatively ample (here, F is an effective divisor), in particular, the relatively effective divisors are *not* effective. Our divisor D_0 will in practice to be something close to the form $-\varepsilon F$ where ε is a small negative number (actually, we may twist by a Cartier divisor, really a test element, from $\operatorname{Spec} R_0$ as well)

Thus, in our situation $\lceil D_p \rceil$

Here is the proof idea. Choose $d^n \in R$ to be a test element for ω_{R_p} (we can essentially find one of these in charcteristic zero if we are clever).

For an appropriate $D = \varepsilon(-F - \operatorname{div}(d^n))$ as above, we construct a diagram:

$$\begin{array}{ccc} \pi_* F^e_* \omega_{X_p}(\lceil p^e D_p \rceil) & \longrightarrow & \pi_* \omega_{X_p}(\lceil D_p \rceil) \cong \pi_* \omega_{X_p} \\ \downarrow & & \downarrow \\ F^e_* d^n \omega_{R_p} & \xrightarrow{\Phi_{R_p}} & \omega_{R_p} \end{array}$$

We know that $\Phi_{R_p}(F^e_* d \omega_{R_p})$ is contained in $\tau(\omega_{R_p}, \Phi_{R_P})$. Thus we have

$$\omega_R = \pi_* \omega_{X_p} = \pi_* \Phi_X(F^e_* \omega_{X_p}(\lceil p^e D_p \rceil)) \subseteq \Phi_R(F^e_* d \omega_{R_p}) \subseteq \omega_{R_p}$$

which completes the proof.

Let us explain how we find our $d \in R_0$ in characteristic zero. We fix d such that $(R_0)_d$ is regular. It follows that some power of d is a test element for $(\omega_{R_p}, \Phi_{R_p})$ in any characteristic. In particular, we may then choose our resolution of singularities $\pi : X_0 \to \operatorname{Spec} R_0$ such that

X_0 is a log resolution of $(X, (d)^n)$ for any integer $n > 0$. We choose $-F$ as above and set $D = \varepsilon(-F - \mathrm{div}(d))$ where $\lceil D \rceil = 0$. After reducing to characteristic $p \gg 0$, find $n > 0$ such that d^n is a test element. Fix p^e such that $\varepsilon p^e \geq n$. We then claim that we have a map $\pi_* \omega_{X_p}(\lceil p^e D_p \rceil) \subseteq d\omega_R$. It is sufficient to check this in codimension 1, and so we are simply reduced to verifying that $\lceil -\varepsilon p^e \div_R (d) \lceil \leq -n \div_R (d)$ which is obvious. The proof of the theorem then follows from the result above.

In fact, the same proof gives us the more general result.

Theorem 2.3. [Har05], [Smi00] *With the notation as above* $(\pi_* \omega_X)_p = \tau(\omega_{R_p}, \Phi_{R_p})$.

Remark 2.4. This was not obvious when it was first proved. While the proof I gave is philosophically the same, it is substantially streamlined in comparison to Hara's original proof. In particular, we avoid several applications of local duality.

Remark 2.5. For a (quasi-)Gorenstein ring R with $R \cong \omega_R$ and $\pi : X \to \operatorname{Spec} R$ as above, $\pi_* \omega_X$ is an example of a multiplier ideal (it is independent of the resolution by [GR70]). Thus the previous result says that the multiplier ideal coincides with the test ideal $\tau(R, \Phi_R)$ for quasi-Gorenstein rings.

Question 2.6. Is it true that if X has Du Bois singularities, then X has dense F-injective type?

Note that X cannot have open F-injective type by the example $k[x, y, z]/(x^3 + y^3 + z^3)$ which is F-injective if and only if $p = 1 \mod 3$.

3. Multiplier ideals, log terminal and log canonical singularities

In the past section, we found analogs of F-injective and F-rational singularities. We want to do the same for F-split singularities.

Definition 3.1. A *pair* (X, Δ) is the combined information of a normal variety X and a (usually effective) $\mathbb{Q}$-divisor Δ. We also typically assume that (X, Δ) is *log* $\mathbb{Q}$*-Gorenstein* which means that $K_X + \Delta \sim_{\mathfrak{q}} mD$ where $m \in \mathbb{Q}$ and D is a Cartier divisor (in other words, this means that $K_X + \Delta$ is $\mathbb{Q}$-Cartier). Occasionally we will also consider *triples* $(X, \Delta, \mathfrak{a}^t)$ where $\mathfrak{a}$ is an ideal sheaf on X and $t \geq 0$ is a real number (additional generalizations are also possible where $\mathfrak{a}^t$ is replaced by a graded system of ideals, or even a formal product of such ideals). For the moment, we will assume that $\mathfrak{a}$ is a

Definition 3.2. A *log resolution* $\pi : \widetilde{X} \to X$ of a pair or triple $(X, \Delta, \mathfrak{a}^t)$ is a resolution of singularities such that $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-G)$ and also with divisorial exceptional set E such that E, G and the strict transform $\pi_*^{-1}\Delta$ of Δ are all in simple normal crossings.

In this setting, we can choose divisors $K_{\widetilde{X}}$ and K_X that agree wherever π is an isomorphism. Then we can consider:

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG = \sum a_i E_i$$

or equivalently

$$K_{\widetilde{X}} + (-\sum a_i E_i) = \pi^*(K_X + \Delta) - tG$$

where the E_i are prime divisors. Here most of the E_i are effective except for those that agree with components of Δ or divisorial components of $V(\mathfrak{a})$. We should explain the term

$\pi^*(K_X + \Delta)$ and note that for the purposes of this course, we will only define this when $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Set choose $0 \neq n \in \mathbb{Z}$ such that $n(K_X + \Delta)$ is Cartier. Then

$$\pi^*(K_X + \Delta) := \frac{1}{n}\pi^*\left(n(K_X + \Delta)\right)$$

The a_i that appear in the above formula are called *discrepancies.* Numbers a_i associated to an exceptional divisor E_i are called *exceptional discrepancies.*

Why might one want to do this (work with these Δ at all)?

(a) If K_X is Cartier (or $\mathbb{Q}$-Cartier, then you can pull back K_X as described above). But if not, it's much less clear how to pull back K_X, see [DH09].

(b) As one changes from one variety to another (via restriction, finite or birational maps) one can pick up a Δ even if you didn't already start with one. For example, if $\pi_* : Y \to X = \operatorname{Spec} k[x, y, z]/(x^4 + y^4 + z^4)$ is the obvious resolution of singularities, then $\pi^* K_X = K_Y + 2E$ where E is the copy of the exceptional divisor. For some purposes, it is useful to keep this information around. In particular, the data of the pair $(Y, 2E)$ may be as good as the data of X.

(c) (This is another variant of (b)) If one is compactifying a variety X, one often compactifies with a nice divisor D such that $\overline{X} \setminus D = X$. Keeping track of this D is also useful.

One can actually define some additional classes of singularities in this setting.

Definition 3.3.

- We say a triple $(X, \Delta, \mathfrak{a}^t)$ is *log canonical* (or *lc*) if all the discrepancies a_i satisfy $a_i \geq -1$. One can check this on a single log resolution.
- We say a triple $(X, \Delta, \mathfrak{a}^t)$ is *Kawamata log terminal* (or *klt*) if all the discrepancies a_i satisfy $a_i \geq -1$. One can check this on a single log resolution.
- We say that a triple $(X, \Delta, \mathfrak{a}^t)$ is *purely log terminal* (or *plt*) if all the exceptional discrepancies a_i satisfy $a_i \geq -1$ for all log resolutions. One needs a sufficiently big log resolution in order to check this.
- One can also define *canonical* and *terminal* singularities by requiring that all exceptional discrepancies satisfy $a_i \geq 0$ and $a_i > 0$ respectively.

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