F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/21-2010

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1. F-rationality

Definition 1.1. Given (M, ϕ) as above, the module $\tau(M, \phi)$ is called the *test submodule of* (M, ϕ). With $\Psi_R : F_*\omega_R \to \omega_R$, the module $\tau(\omega_R, \Psi_R) = \tau(\omega_R)$ is called the simply the *test submodule*. An element $0 \neq d \in R$ is called a *test element for* (M, ϕ) if $dM \subseteq N$ for every nonzero submodule N of M satisfying $\phi(N) \nsubseteq N$. It follows from the above proof that $c \in R$ is such that R_c is regular and $M_c \cong R_c$, then c has some power which is a test element.

If R is a ring of characteristic $p > 0$ and $\pi : \widetilde{X} \to X = \operatorname{Spec} R$ is a resolution of singularities, then philosophically, $\tau(\omega_R)$ should be the submodule corresponding to $\pi_*\omega_{\widetilde{X}}$ (this submodule is independent of the choice of resolution as pointed out in [GR70]). In particular, the same argument we use to prove that F-rational singularities were pseudo-rational, can be used to show that there is always a containment $\tau(\omega_R) \subseteq \pi_*\omega_{\widetilde{X}}$, simply consider the diagram:

$$\begin{array}{ccc}
\pi_*F_*\omega_{\widetilde{X}} = F_*\pi_*\omega_{\widetilde{X}} & \xrightarrow{\pi_*\Psi_{\widetilde{X}}} & \pi_*\omega_{\widetilde{X}} \\
{\scriptstyle F_*\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\
F_*\omega_X & \xrightarrow[\Psi_X]{} & \omega_X
\end{array}$$

We also have the following useful fact about $\tau(M, \phi)$.

Lemma 1.2. *With $\tau(M, \phi)$ as above, $\phi(F^e_*\tau(M, \phi)) = \tau(M, \phi)$.*

Proof. Because ϕ is not zero, $\phi(F^e_*\tau(M, \phi))$ is non-zero. On the other hand, it is clearly ϕ-stable thus $\phi(F^e_*\tau(M, \phi)) \supseteq \tau(M, \phi)$ by the universal property of $\tau(M, \phi)$. However, $\phi(F^e_*\tau(M, \phi)) \subseteq \tau(M, \phi)$ by definition. $\square$

Corollary 1.3. [Vél95] *Suppose that M is a generically rank-1 module, $\phi : F^e_*M \to M$ is R-linear and that $\tau(M, \phi) = M$. Then for any non-zero submodule $N \subseteq M$, there exists an $n > 0$ such that*

$$\phi^n(F^{ne}_*N) = M.$$

In particular, for every non-zero $c \in R$, there exists an $n > 0$ such that $\phi^n(F^{ne}_*cN) = M$.

Proof. Choose $c \in R$ such that $cM \subseteq N$. We may thus assume that $N = cM$. We will show that $\phi^n(F^{ne}_*cM) \subseteq \phi^n(F^{ne}_*cM)$ which will complete the proof since we already know that $\sum_{n>0} \phi^n(F^{ne}_*cM) = M$. Now,

$$\phi^n(F^{ne}_*cM) = \phi^n(F^{ne}_*c\phi(F^e_*M)) = \phi^n(F^{ne}_*F^e_*(c^{p^e}M)) = \phi^{n+1}(F^{(n+1)e}_*c^{p^e}M) \subseteq \phi^{n+1}(F^{(n+1)e}_*cM)$$

as desired. $\square$

Corollary 1.4. *In an F-finite ring, the F-rational locus is open.*

Remark 1.5. I only point this out because using the historically standard definitions, this is much less obvious.

Remark 1.6. The condition of the corollary is sometimes called *strong F-rationality.*

We now try to show that F-rational singularities deform (even though we don't expect pseudo-rational singularities to deform, a problem which I believe is open in general).

Theorem 1.7. *Suppose that R is a reduced local ring and $f \in R$ is a regular element. If R/f has F-rational singularities, then R also has F-rational singularities.*

Proof. The fact that R/f is normal and Cohen-Macaulay immediately imply that R is normal and Cohen-Macaulay. Therefore, we simply have to show that $\tau(\omega_R) = \omega_R$. Choose $c \in R$ such that c is a test element for (ω_R, Ψ_R), and also for $(\omega_{R/f}, \Psi_{R/f})$.

Consider the following diagram of short exact sequences (for every $e > 0$):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & R & \xrightarrow{\times f} & R & \longrightarrow & R/f & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle 1\mapsto cf^{p^e-1}} & & \Big\downarrow{\scriptstyle 1\mapsto c} & & \Big\downarrow{\scriptstyle 1\mapsto \bar{c}} & & \\
0 & \longrightarrow & F^e_* R & \xrightarrow{F^e_* \times f} & F^e_* R & \longrightarrow & F^e_*(R/f) & \longrightarrow & 0.
\end{array}$$

Apply the functor $\mathrm{Hom}_R(\underline{}, \omega_R)$ and note that we obtain the following diagram of short exact sequences.

$$\begin{array}{ccccccccc}
 & & C & \xrightarrow{\eta} & D & \longrightarrow & 0 & & \\
 & & \Big\uparrow & & \Big\uparrow & & \Big\uparrow & & \\
0 & \longrightarrow & \omega_R & \xrightarrow{\times f} & \omega_R & \longrightarrow & \omega_{R/f} \cong \mathrm{Ext}^1_R(R/f, \omega_R) & \longrightarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \alpha} & & \Big\uparrow{\scriptstyle \beta} & & \Big\uparrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & \oplus_e F^e_* \omega_R & \xrightarrow{F^e_* \times f} & \oplus_e F^e_* \omega_R & \longrightarrow & \oplus_e F^e_* \omega_{R/f} \cong \oplus_e \mathrm{Ext}^1_R(F^e_*(R/f), \omega_R) & \longrightarrow & 0
\end{array}$$

where α is the dual map to the map $R \to F^e_* R$ that sends 1 to c, and β is the dual map to the map which sends 1 to cf^{p^e-1}. Sticking direct sums in from of the terms in the bottom row guarantees that the image of α is $\tau(\omega_R)$ and that the image of δ is $\tau_R(\omega_{R/f}) = \omega_{R/f}$ by hypothesis. Of course, the image of β is contained in $\tau_R(\omega_R)$. Thus D has a natural surjection onto $C = \omega_R/\tau(\omega_R)$. Furthermore, the composition $C \to D \to C$ is as before, multiplication by f and Nakayama's lemma implies that C is zero again. $\square$

Finally, let's also compare some of the other basic properties of F-rational singularities with those of rational singularities. In particular, we might ask if Boutot's theorem still holds?

Theorem 1.8. *Suppose that $i : R \to S$ is a finite inclusion of normal local domains that splits. Then if S is F-rational (respectively F-injective) then R is F-rational (respectively F-injective).*

Proof. We first show that R is Cohen-Macaulay (note that in either case, S is Cohen-Macaulay by hypothesis). Set $\kappa : S \to R$ to be the splitting of i. By dualizing the composition $\kappa \circ i : R \to S \to R$, we obtain

$$\begin{array}{ccccc} R\operatorname{Hom}_R(R, \omega_R^\bullet) & \longleftarrow & R\operatorname{Hom}_R(S, \omega_R^\bullet) & \longleftarrow & R\operatorname{Hom}_R(R, \omega_R^\bullet) \\ \| & & \| & & \| \\ \omega_R^\bullet & \longleftarrow & \omega_S^\bullet & \longleftarrow & \omega_R^\bullet \end{array}$$

Just as in the original Boutot's theorem, we immediately obtain that R is Cohen-Macaulay since the identity $h^{-\dim R+i}\omega_R^\bullet \to h^{-\dim R+i}\omega_R^\bullet$ factors through zero for $i > 0$.

For the F-injectivity, we have things pretty easy. We know that the natural map $\omega_S \to \omega_R$ is surjective. But we also have the diagram:

$$\begin{array}{ccc} \omega_R & \twoheadleftarrow & \omega_S \\ {\scriptstyle \Psi_R}\uparrow & & \uparrow\!\!\uparrow {\scriptstyle \Psi_S} \\ F^e_*\omega_R & \twoheadleftarrow & F^e_*\omega_S \end{array}$$

Since Ψ_S is surjective, Ψ_R is also surjective which implies that R is F-injective.

For F-rationality, the argument is very very similar. We write down essentially the same diagram.

$$\begin{array}{ccc} \omega_R & \longleftarrow & \omega_S \\ {\scriptstyle \alpha}\uparrow & & \uparrow {\scriptstyle \beta} \\ \oplus_e F^e_*\omega_R & \longleftarrow & \oplus_e F^e_*\omega_S \end{array}$$

Now however, the maps α and β are Ψ_R and Ψ_S (respectively) pre-multiplied by some element $c \in R$ that is a test element for both ω_R and ω_S. As before, β is surjective which implies that α is surjective. ☐

Remark 1.9. Without the condition that S is a *finite* extension of R, these results are false. See [Wat97].

2. Reduction to Characteristic p

Our goal over the next couple weeks is to give a proof that F-rational singularities correspond to rational singularities via reduction mod p. This is hard. We will break this up into several steps.

- Introduce reduction to characteristic $p \gg 0$.
- Modulo a really hard technical lemma, prove the theorem.
- Prove the really hard technical lemma (we might put this off a little bit).

In this section we go over the necessary prerequisites to reduce a variety to characteristic p. A good introductory reference to this theory is [HH06, 2.1]. Our primary goal is the statements needed to work with rational singularities.

Let R be a finitely generated $\mathbb{C}$ algebra. We can write $R = \mathbb{C}[x_1, \dots, x_n]/I$ for some ideal I and let S denote $\mathbb{C}[x_1, \dots, x_n]$. Let $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. Let $\pi : Bl_J(Y) = \widetilde{Y} \to Y$ be a strong (projective) log resolution of X in Y with reduced exceptional divisor E mapping to X (induced by blowing up an ideal J). Note that we may also assume our schemes are projective; that is, we can embed Y as an open set in some $\mathbb{P}^n$, and thus take the projective closure $\overline{X}$ of X in $\mathbb{P}^n$. We may even extend π (our embedded resolution) to $\overline{\pi} : \widetilde{\mathbb{P}^n} \to \mathbb{P}^n$, a strong (projective) log resolution of $\overline{X}$ with reduced exceptional set $\overline{E}$.

There exists a finitely generated $\mathbb{Z}$ algebra $A \subset \mathbb{C}$ (including all the coefficients of a set of generators of I and those required by the blow-up of J), a finitely generated A algebra R_A, an ideal $J_A \subset R_A$, and schemes $\widetilde{Y}_A$, $\overline{X}_A$ and E_A of finite type over A such that $R_A \otimes_A \mathbb{C} = R$, $J_A R = J$, $\overline{X}_A \times_{\operatorname{Spec} A} \mathbb{C} = \overline{X}$, $Y_A \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C} = Y$, $\overline{E}_A \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C} = \overline{E}$ and $E_A \times_{\operatorname{Spec} A} \operatorname{Spec} \mathbb{C} = E$ with E_A effective and supported on the blow-up of J_A. We may localize A at a single element so that Y_A is smooth over A and E_A is a simple normal crossing divisor over A if desired. By further localizing A (at a single element), we may assume any finite set of finitely generated R_A modules is A-free, see for example [Hun96, 3.4] and [HR74] and we may assume that A itself is regular. We can also take any finite collection of modules, for example $R^i f_* \mathcal{O}_X$ to this mixed characteristic setting, as well as maps between these modules.

Theorem 2.1 (Generic Freeness)**.** [HR74] *Let A be a Noetherian domain and let R be a finitely generated A-algebra. Let S be a finitely generated R-algebra and let E be a finitely generated S-module. Let M be a finited generated R-submodule E and let N be a be a finitely generated A-submodule. Let $D = E/(M + N)$. Then there is a nonzero element $a \in A$ such that D_a is a free A_a-module.*

In our particular case, we may localize so that S_A, R_A, I_A, J_A, etc. are all locally free over A, as well as the various cokernels of maps between these modules.

We will now form a family of positive characteristic models of X by looking at all the rings $R_t = R_A \otimes_A k(t)$ where $k(t)$ is the residue field of a maximal ideal $t \in T = \operatorname{Spec} A$. Note that $k(t)$ is a finite, and thus perfect, field of characteristic p. In the case where we are reducing a particular maximal (closed) point, tensoring with $k(t)$ will either give us a unique closed point in our characteristic p model (if we started over $\mathbb{C}$ as we assumed), or a possibly finite set of closed points if we began by working over some other field of characteristic zero. If we are working with a non-closed point, we will have a finite set of points of $\operatorname{Spec} R_t$ pulling back to x_A. We may also tensor the various schemes Y_A, E_A, etc. with $k(t)$ to produce a characteristic p model of an entire situation.

Example 2.2. If we let $R = \mathbb{C}[x, y, z]/(x^2 + y^2 + z^2)$, then we would let $A = \mathbb{Z}$, so that $S_A = A[x, y, z]$, $R_A = S_A/(x^2 + y^2 + z^2)$, $X_A = \operatorname{Spec} R_A$, and $Y_A = \operatorname{Spec} S_A$. An obvious resolution is just blowing up the point (x, y, z) so that is what we do in S_A as well to get $\pi_A : (\widetilde{Y}_A = \operatorname{Proj}(S_A \oplus (x, y, z)t \oplus (x, y, z)^2 t^2 \oplus \ldots)) \to Y_A$. In characteristic 2, this resolution is *not* a resolution of singularities since $X_{\mathbb{Z}/2}$ isn't even reduced! However, in all other characteristics it is.

Various properties of rings that we are interested in descend well from characteristic zero. For example, smoothness, normality, being reduced, and being Cohen-Macaulay all descend well [Hun96, Appendix 1]. Specifically, R_t has one of the above properties above for an open set of maximal ideals of A if and only if $R_{(\operatorname{Frac} A)}$ has the same property (in which case so does R). Furthermore, a ring R of finite type over a field k is Cohen-Macaulay if and only if for every field extension $k \subset K$, $R \otimes_k K$ is Cohen-Macaulay [BH93, 2.1.10]. Thus R_t is Cohen-Macaulay for an infinite set of primes if and only if R is Cohen-Macaulay. Likewise, it has already been shown that if R_t is seminormal for a Zariski dense set of primes, then R is seminormal [HR76, 5.31].

Let us show that the Cohen-Macaulay condition descends to characteristic $p > 0$.

Example 2.3. Suppose that $R = S/I$ where $S = \mathbb{C}[x_1, \ldots, x_n]$. We know that $R\operatorname{Hom}_S(R/I, S) \cong \omega_R^{\bullet}[-\dim S]$. We consider $h^i(R\operatorname{Hom}_S(R, S))$ for some i. We will show that this vanishes in characteristic zero if and only if it vanishes in characteristic $p \gg 0$. Choose A to be a finitely generated $\mathbb{Z}$-algebra containing all the coefficients of a set of generators $\{f_i\}$of I. Let $S_A = A[x_1, \ldots, x_n]$ and set I_A to be the ideal in S_A generated by those same $\{f_i\}$. Set $R_A = S_A/I_A$. If necessary, we replace A by a localization such that all modules in sight are A-free.

We claim first that $h^i(R\operatorname{Hom}_S(R, S)) \cong h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \otimes_A \mathbb{C}$. But this is easy, since it is the same thing as $h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \otimes_{S_A} S$ noticing that S is a flat S_A-algebra (see for example, [**?**, Theorem 7.11, Exercise 7.7]). Therefore, we have that $h^i(R\operatorname{Hom}_S(R, S)) \neq 0$ if and only if $h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \neq 0$ because the latter term is A-free.

We choose $\mathfrak{p}$ to be a maximal ideal of $\mathbb{A}$ and we want to do the same thing base-changing with $k = A/\mathfrak{p}$. In particular, we need to show that

$$h^i(R\operatorname{Hom}_{S_A}(R_A, S_A)) \otimes_A k \cong h^i(R\operatorname{Hom}_{S_k}(R_k, S_k)).$$

This is more complicated because k is not A-flat. Choose a free S_k-resolution $P_{\bullet}$ of R_A, tensoring with S_k over S_A turns it into a complex mapping to R_k. Alternately, tensoring with k over A keeps it acyclic (since it would then correspond to Tor of the A-free module R_A). Thus, it is still a free-resolution of R_k. The statement then reduces to the question of whether $\operatorname{Hom}_{S_A}(\_\_, S_A) \otimes k$ is the same as $\operatorname{Hom}_{S_A}(\_\_ \otimes_A k, S_k)$ for A-free modules in the blank. Choose M to fill in the blank, an A-free S_A-module. Choose $F \to G \to M \to 0$ to be an exact sequence with F and G chosen as S_A-free modules, by localizing further, we may assume that $H = \operatorname{Image}(F) \subseteq G$ is A-free. We have a natural map

$$\gamma(F) : \operatorname{Hom}_{S_A}(F, S_A) \otimes k \to \operatorname{Hom}_{S_A}(F \otimes_A k, S_k)$$

This can also be described as

$$\operatorname{Hom}_{S_A}(F, S_A) \otimes_{S_A} S_k \to \operatorname{Hom}_{S_A}(F \otimes_{S_A} S_k, S_k)$$

But since $= S_A^n$, this is just $S_k^n \to S_k^n$ in an obvious isomorphism. Thus $\gamma(F)$ and $\gamma(G)$ are isomorphisms. Now, consider the following diagram (where all tensor products are over A)

$$\begin{array}{ccccccc}
\operatorname{Tor}_A^1(\operatorname{Hom}_{S_A}(H, S_A), k) = 0 & \to & \operatorname{Hom} S_A(M, S_A) \otimes k & \to & \operatorname{Hom}_{S_A}(G, S_A) \otimes k & \to & \operatorname{Hom} S_A(F, S_A) \otimes k \\
\downarrow & & \downarrow & & \downarrow \sim & & \downarrow \sim \\
\operatorname{Hom}_{S_A}(\operatorname{Tor}_A^1(H, k), S_k) = 0 & \longrightarrow & \operatorname{Hom}_{S_A}(M \otimes k, S_k) & \longrightarrow & \operatorname{Hom}_{S_A}(G \otimes k, S_k) & \longrightarrow & \operatorname{Hom}_{S_A}(F \otimes k, S_k)
\end{array}$$

Therefore, if you want to determine if a ring is Cohen-Macaulay, in some sense it is sufficient to check it in characteristic $p \gg 0$.

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