F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/21-2010

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1. F-rationality

First we do an example we didn't finish last time.

Example 1.1. Let E be an ordinary elliptic curve (we know this it is F-split) and suppose that $X = E \times_k \mathbb{P}^1$ is the trivial ruled surface over E. Let S be a section ring of X with respect to a (very) very ample divisor. We will show that S is F-split (equivalently, that X is F-split) but that S is not Cohen-Macaulay. First we show that S is not Cohen-Macaulay. It is enough to show that $H^2_{S_+}(S) \neq 0$. But, $(H^2_{S_+}(S))_0 = H^1(X, \mathcal{O}_X)$. By [Har77, Chapter V, Lemma 2.4] (basic facts about the Cohomology of ruled surfaces) imply that this is $H^1(E, \mathcal{O}_E) \neq 0$ because E is an elliptic curve. Now we need to show that X is F-split. This follows from the following easy lemma:

Lemma 1.2. *Suppose that X and Y are Frobenius split schemes of finite type over k. Then $X \times_k Y$ is also Frobenius split.*

Proof. Choose $\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$ and $\psi : F_*\mathcal{O}_Y \to \mathcal{O}_Y$ both splittings (in other words, sends 1 to 1). We will construct a splitting on $X \times_k Y$. We will do it locally (but canonically) so that the splitting clearly glues. Thus assume that $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$. We need to construct a splitting of the Frobenius map $F_{R\otimes_k S} : R \otimes_k S \to F_*R \otimes_k S$. Given $r \otimes s \in R \otimes_k S$, we define $\alpha(r \otimes s) = \phi(r) \otimes \psi(s)$. This map is obviously $R \otimes S$-linear, and it sends 1 to 1, it also clearly glues. □

Because of this, Fedder suggested that normal, Cohen-Macaulay and F-injective might be a closer match to rational singularities than F-purity. There was some evidence for this. In particular, Fedder showed that certain classes of hypersurfaces (defined over $\mathbb{Z}$) had rational singularities over $\mathbb{C}$ if and only if for all sufficiently large $p > 0$, the singularity viewed modulo p had F-pure (equivalently, F-injective) singularities. Notice that this doesn't allow $x^3+y^3+z^3$ because that does not have F-pure singularities for $p = 2 \bmod 3$. Elkies has since shown that for cones over planar elliptic curves (none of which have rational singularities), they are supersingular (and thus ordinary) for infinitely many p. If you are considering cones over Calabi-Yau varieties (for simplicity, we also assume that these cones are also Cohen-Macaulay, for example a K3-surface), then the condition that $\phi : F^e_*\omega_S \to \omega_S$ is known for surfaces and open for higher dimensional varieties.

F-injective singularities still aren't quite good enough. Consider the following attempted proof at showing the Cohen-Macaulay F-injective singularities are rational (ignoring the issue of characteristic $p > 0$ reduction for now).

Not a proof. Given a resolution of singularities $\pi : \widetilde{X} \to X = \operatorname{Spec} R$, we want to show that $\pi_* \omega_{\widetilde{X}} = \omega_X$. Consider the diagram:

$$\begin{array}{ccc} \pi_* F_* \omega_{\widetilde{X}} = F_* \pi_* \omega_{\widetilde{X}} & \xrightarrow{\pi_* \Psi_{\widetilde{X}}} & \pi_* \omega_{\widetilde{X}} \\ {\scriptstyle F_* \alpha} \big\downarrow & & \big\downarrow {\scriptstyle \alpha} \\ F_* \omega_X & \xrightarrow[\Psi_X]{} & \omega_X \end{array}$$

where the horizontal maps are the natural maps dual to Frobenius. If one can show that $\pi_* \Psi_{\widetilde{X}}$ and α are surjective, then that would imply that Ψ_X is surjective. Going the other way seems hard though. The following definition was thus given which easily implies that α is surjective. □

Definition 1.3. An F-finite reduced ring R is called *F-rational* if it is Cohen-Macaulay and there are no proper / non-zero submodules of ω_X stable under Ψ_X (ie, $M \subseteq \omega_X$ such that $\Psi_X(M) \subseteq M$).

Why is this definition motivated? Well, in a polynomial ring with $X = \operatorname{Spec} k[x_1, \ldots, x_n]$, Φ_X^e can be identified with the map $F_*^e k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_n]$ that sends $x_1^{p^e-1} \ldots x_n^{p^e-1}$ to 1 and all the other monomials to zero. Given any polynomial $f \in k[x_1, \ldots, x_n]$, we can always find a monomials m and an $e \gg 0$ such that $\Phi_X^e(mf) = 1$. Thus, there are no Φ_X-stable proper ideals in a polynomial ring.

Definition 1.4. [LT81] $X = \operatorname{Spec} R$ is said to have pseudo-rational singularities if it is Cohen-Macaulay and also for every proper birational map $\pi : \widetilde{X} \to X$ with $\widetilde{X}$ normal, $\pi_* \omega_{\widetilde{X}} = \omega_X$.

Remark 1.5. If R does not necessarily have a dualizing complex, then another definition is used (using local cohomology modules instead of ω_X, this is tantamount with replacing R by its completion). Lipman proved that regular rings have rational singularities (and that this holds under extreme generality).

Theorem 1.6 (Smith)**.** *If R is F-rational, then R is pseudo-rational.*

Proof. This should be immediate from the diagram above. □

We will show that F-rational singularities satisfy many nice properties. In particular, we will study their deformations, how they behave under summands, etc. We will also show that F-rational singularities really do coincide with rational singularities by reduction mod $p > 0$.

We have defined 3 different classes of singularities now. F-rational, F-split, and F-injective (the last one has both Cohen-Macaulay and non-Cohen-Macaulay variants). We also know that F-rational singularities are F-injective (and Cohen-Macaulay) and that F-pure singularities are F-injective (meaning $h^i(F_* \omega_R^\bullet) \to h^i(\omega_R^\bullet)$ surjects for all $i > 0$, or dually $H_{\mathfrak{m}}^i(\mathcal{O}_X) \to H_{\mathfrak{m}}^i(F_* \mathcal{O}_X)$ injects for all $i > 0$). We will now investigate the normality properties of F-injective and F-rational singularities.

Lemma 1.7. *Suppose that R is F-finite and F-rational, then R is normal.*

Proof. Without loss of generality, we may assume that R is local. Let R^N be the normalization of R. We have the following inclusion map $i : R \to R^N$. We will prove that the map is an isomorphism. R is already Cohen-Macaulay, and so it is S2, and so it by Serre's criterion for normality, we simply need to check that R is regular in codimension 1. Thus by localizing we can assume that R is a 1-dimensional ring (and thus so is R^N, which is now regular). We have the following diagram of rings.

$$\begin{array}{ccc} R^N & \xrightarrow{F_{R^N}} & F_* R^N \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle F_* i} \\ R & \xrightarrow[F_R]{} & F_* R \end{array}$$

Apply $R\operatorname{Hom}_R(\_\_, \omega_R^{\bullet})$, and then Grothendieck duality for a finite map i gives us the following dual diagram.

$$\begin{array}{ccc} \omega^{\bullet}_{R^N} & \longleftarrow & F_*\omega^{\bullet}_{R^N} \\ {\scriptstyle i^{\vee}}\downarrow & & \downarrow{\scriptstyle F_* i^{\vee}} \\ \omega^{\bullet}_R & \longleftarrow & F_*\omega^{\bullet}_R \end{array}$$

All the rings in question are Cohen-Macaulay, so we can remove all the dots and merely work with sheaves. We simply need to show that $i^{\vee}$ is injective because an isomorphism of the induced map of dualizing complexes, will imply that the original map was an isomorphism. Now, if W is a the multiplicative system of elements not contained in any minimal prime of R, we also have the diagram

$$\begin{array}{ccc} \omega_{R^N} & \xrightarrow{\gamma} & W^{-1}(\omega_{R^N}) \cong K(R) \\ {\scriptstyle i^{\vee}}\downarrow & & \| \\ \omega_R & \longrightarrow & W^{-1}(\omega_R) \cong K(R) \end{array}$$

where $K(R)$ is the total field of fractions of R. We notice that ω_{R^N} is torsion-free on each irreducible component thus the map γ is injective which implies that $i^{\vee}$ is also injective. $\square$

Now we turn to F-injectivity, we do not assume that R is Cohen-Macaulay but rather that $H^i_{\mathfrak{m}}(\mathcal{O}_X) \to H^i_{\mathfrak{m}}(F_*\mathcal{O}_X)$ injects for every maximal ideal $\mathfrak{m} \in R$. Note that this condition localizes, in particular $h^i(F_*\omega^{\bullet}_{R_{\mathfrak{q}}}) \to h^i(\omega^{\bullet}_{R_{\mathfrak{q}}})$ surjecting localizes.

Lemma 1.8. *Suppose that (R, m) is a reduced local ring of characteristic p, $X = \operatorname{Spec} R$ and that $X \setminus m$ is weakly normal. Then X is weakly normal if and only if the action of Frobenius is injective on $H^1_m(R)$.*

Proof. We assume that the dimension of R is greater than 0 since the zero-dimensional case is trivial. Embed R in its weak normalization $R \subset R^{\mathrm{WN}}$ (which is of course an isomorphism outside of m). We have the following diagram of R-modules.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R & \hookrightarrow & \Gamma(X \setminus m, \mathcal{O}_{X-m}) & \twoheadrightarrow & H^1_m(R) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow & & \\ 0 & \longrightarrow & R^{\mathrm{WN}} & \hookrightarrow & \Gamma(X^{\mathrm{wn}} \setminus m, \mathcal{O}_{X^{\mathrm{wn}}-m}) & \twoheadrightarrow & H^1_m(R^{\mathrm{WN}}) & \longrightarrow & 0 \end{array}$$

The left horizontal maps are injective because R and R^{WN} are reduced. One can check that Frobenius is compatible with all of these maps. Now, R is weakly normal if and only if R is weakly normal in R^{WN} if and only if every $r \in R^{\mathrm{WN}}$ with $r^p \in R$ also satisfies $r \in R$ by Proposition **??**.

First assume that the action of Frobenius is injective on $H^1_m(R)$. So suppose that there is such an $r \in R^{\mathrm{WN}}$ with $r^p \in R$. Then r has an image in $\Gamma(X \setminus m, \mathcal{O}_{X-m})$ and therefore an image in $H^1_m(R)$. But r^p has a zero image in $H^1_m(R)$, which means r has zero image in $H^1_m(R)$, which guarantees that $r \in R$ as desired.

Conversely, suppose that R is weakly normal. Let $r \in \Gamma(X \setminus m, \mathcal{O}_{X-m})$ be an element such that the action of Frobenius annihilates its image $\overline{r}$ in $H^1_m(R)$. Since $r \in \Gamma(X \setminus m, \mathcal{O}_{X-m})$ we identify r with a unique element of the total field of fractions of R. On the other hand, $r^p \in R$ so $r \in R^{\mathrm{WN}} = R$. Thus we obtain that $r \in R$ and so $\overline{r}$ is zero as desired. □

Theorem 1.9. *Let R be a reduced F-finite ring with a dualizing complex. If R is F-injective then R is weakly normal (and thus in particular seminormal). Furthermore, R is weakly normal if and only if $H^1_{\mathfrak{q}}(R_{\mathfrak{q}}) \to H^1_{\mathfrak{q}}(F_* R_{\mathfrak{q}})$ injects for all $\mathfrak{q} \in \operatorname{Spec} R$.*

Proof. A ring is weakly normal if and only if all its localizations at prime ideals are weakly normal [RRS96, 6.8]. If R is not weakly normal, choose a prime $P \in \operatorname{Spec} R$ of minimal height with respect to the condition that R_P is not weakly normal. Apply Lemma 1.8 to get a contradiction. □

Corollary 1.10. *If R is a one dimensional F-finite reduced ring, then R is weakly normal if and only if it is F-injective. In particular, if R is local and has perfect residue field, then R is weakly normal if and only if R is F-split.*

This also gives us another example of an F-injective singularity that is not weakly normal.

Example 1.11. The curve singularity corresponding to the pushout $\{\mathbb{F}_p(t)[x] \to \mathbb{F}_p(t)[x]/(x) = \mathbb{F}_p(t) \leftarrow \mathbb{F}_p(t^p)[s]\}$ is weakly normal, but not F-split, since the residue field extension over the singular point (when mapping ot the normalization) is not separable.

We now return to our study of F-rationality. In the case that R is a domain, we will also show that ω_R has a unique smallest submodule stable under Φ_X.

First we need a lemma.

Lemma 1.12. *Suppose that $R \to S$ is a finite map of rings such that $\operatorname{Hom}_R(S, R)$ is isomorphic to S as an S-module. Further suppose that M is a finite S-module.*

Then the natural map

$$\operatorname{Hom}_S(M, S) \times \operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(M, R) \tag{1}$$

induced by composition is surjective.

Proof. First, set α to be a generator (as an S-module) of $\operatorname{Hom}_R(S, R)$. Suppose we are given $f \in \operatorname{Hom}_R(M, R) \cong \operatorname{Hom}_R(M \otimes_S S, R)$. We wish to write it as a composition.

Using adjointness, this f induces an element $\Phi(f) \in \operatorname{Hom}_S(M, \operatorname{Hom}_R(S, R))$. Just as with the usual Hom-Tensor adjointness, we define $\Phi(f)$ by the following rule:

$$(\Phi(f)(t))(s) = f(t \otimes s) = f(st) \text{ for } t \in M,\ s \in S.$$

Therefore, since $\operatorname{Hom}_R(S, R)$ is generated by α, for each f and $t \in M$ as above, we associate a unique element $a_{f,t} \in S$ with the property that $(\Phi(f)(t))(\_\_) = \alpha(a_{f,t}\_\_)$.

Thus using the isomorphism $\mathrm{Hom}_R(S,R) \cong S$, induced by sending α to 1, we obtain a map $\Psi : \mathrm{Hom}_R(M,R) \to \mathrm{Hom}_S(M,S)$ given by $\Psi(f)(t) = a_{f,t}$.

We now consider $\alpha \circ (\Psi(f))$. However,

$$\alpha(\Psi(f)(t)) = \alpha(a_{f,t}) = (\Phi(f)(t))(1) = f(t).$$

Therefore $f = \alpha \circ (\Phi(f))$ and we see that the map (1) is surjective as desired. □

In particular, this yields the following corollary.

Corollary 1.13. *If $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ generates $\mathrm{Hom}_R(F^e_* R, R)$ as an R-module, then ϕ^l generates $\mathrm{Hom}_R(F^{le}_* R, R)$ as an $F^{el}_* R$-module for all $l > 0$.*

Theorem 1.14 (Hochster-Huneke, Blickle-Böckle)**.** *Suppose that R is an F-finite domain and that M is a torsion-free rank one R-module with a non-zero map $\phi : F^e_* M \to M$. Then there exists a unique smallest non-zero submodule $\tau(M,\phi) \subseteq M$ which is stable under ϕ (in other words, which satisfies $\phi(F^e_* N) \subseteq N$).*

Proof. Since ϕ is non-zero and M is rank-1, ϕ is generically surjective. Choose $c \in R$ such that

(i) $\phi_c : F^e_* M_c \to M_c$ generates $\left(\mathrm{Hom}_R(F^e_* M, M)\right)_c$ as an $F^e_* R$-module.
(ii) $cM \subseteq \phi(F^e_* M)$
(iii) $M_c \cong R_c$ and $F^e_* R_c \cong F^e_* M_c$ is a free R_c-module.

Condition (i) is possible because the map of $F^e_* R$-modules

$$\langle \phi \rangle_{F^e_* R} \to \mathrm{Hom}_R(F^e_* M, M)$$

is generically surjective (since ϕ is non-zero) because $\mathrm{Hom}_R(F^e_* M, M)$ is a rank one $F^e_* R$-module. Condition (ii) and (iii) are possible since M is rank-one.

Suppose now that $N \subseteq M$ is a ϕ-stable submodule. Our immediate goal is to show that $N_c = M_c \cong R_c$. Choose a prime $\mathfrak{q} \in \mathrm{Spec}\, R_c$, it is enough to show that $N_\mathfrak{q} = M_\mathfrak{q} \cong R_\mathfrak{q}$. Choose $0 \neq n \in N_\mathfrak{q}$ and choose $l \gg 0$ such that $F^{le}_* n \notin \mathfrak{q} \cdot F^{le}_* M_\mathfrak{q} = F^l_*(\mathfrak{q}^{[p^e]} R_\mathfrak{q})$. By hypothesis, $F^{le}_* M_\mathfrak{q}$ is a free $R_\mathfrak{q}$-module, so that $F^l_*(M_\mathfrak{q}/\mathfrak{q}^{[p^e]})$ is also free as an $R/\mathfrak{q}$-module of the same rank. Choose elements $a_2, \dots a_k \in M_\mathfrak{q}$ such that the images of $a_1 = n, a_2, \dots, a_k$ form a basis for $F^{le}_* M_\mathfrak{q}/\mathfrak{q}^{[p^e]}$ as an $R_\mathfrak{q}/\mathfrak{q}$-module. We have a map $\gamma : \oplus_i a_i R \to F^{\leq}_* M_\mathfrak{q}$.

By Nakayama's lemma, γ is surjective. But it is a surjective map between free modules of the same rank, so it is also injective. Therefore, $a_1, a_2, \dots, a_k$ form a basis for $F^{le}_* M_\mathfrak{q}/\mathfrak{q}^{[p^e]}$ over $M_\mathfrak{q}$. In particular, by projecting onto the first coordinate, there exists a map $\psi : F^{le}_* M_\mathfrak{q} \to M_\mathfrak{q}$ such that $\psi(F^{le}_* n R_\mathfrak{q}) = M_\mathfrak{q}$ (notice that $F^{le}_* n R_\mathfrak{q}$ is not the summand generated by n, but it contains it). Thus $\psi(F^{le}_* N_\mathfrak{q}) = M_\mathfrak{q}$. However, $\psi(\_\_) = \phi^l(d \cdot \_\_)$ by (i) which implies that $M_\mathfrak{q} \supseteq N_\mathfrak{q} \supseteq \phi^l(F^{le}_* N_\mathfrak{q}) = M_\mathfrak{q}$ also.

Because $N_c = M_c$, we know that $c^n M \subseteq N$ for some $n > 0$. We will show that $n = 2$ works. Choose $l \gg 0$ such that $p^{le} \geq n + 1$. Then

$$c^2 M \subseteq c\phi^l(F^{le}_* M) = \phi^l(F^{le}_* c^{p^{le}} M) \subseteq \phi^l(F^{le}_* c^n M) \subseteq \phi^l(F^{le}_* N) \subseteq N$$

as desired. We call the element c^2 a *test element* for (M, ϕ).

Finally, we construct $\tau(M, \phi)$.

$$\tau(M,\phi) := \sum_{l \geq 0} \phi^l(F^{le}_* c^2 M)$$

It is certainly non-zero, and it is contained in any ϕ-stable N by construction. This completes the proof. ☐

References

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