F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/21-2010

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1. Deformations of F-split and rational singularities.

One very fundamental property of rational singularities is the fact that they behave well in families. In fact, one also has the (a-priori) more general statement. We will prove it because eventually we will try to mimic it in characteristic $p > 0$.

First we need a finer version of resolution of singularities.

Definition 1.1. With X a reduced scheme and $Z \subset X$ any scheme (reduced or not), we say that a resolution of singularities $\pi : \widetilde{X} \to X$ is a *log resolution of* $Z \subseteq X$ if in addition we assume.

- (i) $\mathscr{I}_Z \cdot \mathcal{O}_{\widetilde{X}}$ is a invertible sheaf. In other words, it is equal to $\mathcal{O}_{\widetilde{X}}(-G)$.
- (ii) $\operatorname{exc} \pi \cup \operatorname{Supp}(G)$ is a divisor with simple normal crossings.

Remark 1.2. Log resolutions also exist in characteristic zero. Again, they also may be taken to satisfy the following properties.

- (a) π is projective, in other words, it is the blow-up of some (horrible) ideal.
- (b) π is an isomorphism on $X \setminus (\operatorname{Sing}(X) \cup Z)$.
- (c) π is obtained by a sequence of blow-ups at smooth subvarieties (if $X \subseteq Y$ and Y is smooth, one may instead require that π is obtained by a sequence of blow-ups at smooth points of Y).

Theorem 1.3. [Elk78] *Suppose that R is a local ring and that $f \in R$ is a regular element such that R/f has rational singularities, then R also has rational singularities.*

Proof. Note that since R/f is rational, R/f and thus R is Cohen-Macaulay. Let $\pi : \widetilde{X} \to X = \operatorname{Spec} R$ be a resolution of X that is also simultaneously a resolution of $H = \operatorname{Spec} R/f$. Let $\overline{H}$ be the total transform of H (that is, $\overline{H}$ is the scheme defined by $f\mathcal{O}_{\widetilde{X}}$) and let $\widetilde{H}$ denote the strict transform of H. Note, there is a natural inclusion of schemes $\widetilde{H} \to \overline{H}$. Consider the following diagram.

$$\begin{tikzcd}
 & & & \pi_*\omega_{\widetilde{H}} \arrow[d] & \\
0 \arrow[r] & \pi_*\omega_{\widetilde{X}} \arrow[r, "\times f"] \arrow[d, hook, "\psi"] & \pi_*\omega_{\widetilde{X}} \arrow[r] \arrow[d, hook, "\psi"] & \pi_*\omega_{\overline{H}} \arrow[r] \arrow[d, two heads, "\phi"] & 0 \\
0 \arrow[r] & \omega_X \arrow[r, "\times f"] & \omega_X \arrow[r] & \omega_H \arrow[r] & 0
\end{tikzcd}$$

The bottom row is exact because H is Cohen-Macaulay. The top row is exact by Grauert-Riemenschneider vanishing, [GR70]. The map labeled ϕ is surjective since the vertical composition from $\pi_*\omega_{\widetilde{H}}$ is an isomorphism. It is then enough to show that ψ is surjective.

Let C be the cokernel of ψ. The fact that ϕ is surjective means that $C \xrightarrow{\times f} C$ is surjective by the snake lemma. But this contradicts Nakayama's lemma, completing the proof. $\square$

One can ask the slight different (a priori) question of whether rational singularities actually deform in families. In other words, given a flat family $X \to C$ over a smooth curve C, such that one fiber has rational singularities, do the nearby fibers also have rational singularities? In order to answer this, we will first need a lemma.

Lemma 1.4. *Suppose that $X/k = \bar{k}$ has rational singularities and $H \subseteq X$ is a general member of a base-point free linear system (or is simply defined by a sufficiently general equation) on X. Then H also has rational singularities.*

Proof. Let $\pi : \widetilde{X} \to X$ be a resolution of singularities. Let $\widetilde{H}$ denote the strict transform of H (because H is general, $\widetilde{H} = \pi^{-1}(H)$). Since the linear system on X lifts to a base-point free linear system on $\widetilde{X}$, $\widetilde{H}$ is a general member of a base-point-free linear system on $\widetilde{X}$ and thus it is smooth. We will show that $R\pi_* \mathcal{O}_{\widetilde{H}} = \mathcal{O}_H$.

We work locally and assume that $H = V(f)$ for some $f \in R$ where $X = \operatorname{Spec} R$. We first claim that $L\pi^* \mathcal{O}_H \cong \pi^* \mathcal{O}_H = \mathcal{O}_{\widetilde{H}}$, but this is easy since we have the short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

Since the first two terms have trivial $L^i f^*$ for $i > 0$, so does $\mathcal{O}_H = R/f$. Thus,

$$R\pi_* \mathcal{O}_{\widetilde{H}} \cong R\pi_* L\pi^* \mathcal{O}_H \cong R\pi_* (\mathcal{O}_{\widetilde{X}} \underline{\underline{\otimes}} L\pi^* \mathcal{O}_H) \cong (R\pi_* \mathcal{O}_{\widetilde{X}}) \underline{\underline{\otimes}} \mathcal{O}_H \cong \mathcal{O}_X \underline{\underline{\otimes}} \mathcal{O}_H \cong \mathcal{O}_H,$$

as desired. $\square$

Corollary 1.5. *If $f : X \to C$ is a proper family over a curve C and a fiber $f^{-1}(c)$ has rational singularities, then so do the nearby fibers.*

Proof. By Elkik's result, X has rational singularities near $f^{-1}(c)$. Choose an open set $U \subseteq X$ containing $f^{-1}(c)$ to be such that U has rational singularities. Let $Z = X \setminus U$. Then $f(Z)$ is a set of points of C (it is closed, and doesn't contain $c \in C$). A general element of C will give a general fiber of X and that fiber will be a general element of U. Thus that fiber will have rational singularities. $\square$

In this section, we'll point out that F-split singularities need not be normal, or Cohen-Macaulay (even when they are normal). We'll also show that they don't deform. It is this failure of deformation that will lead us to the right variant of rational singularities in characteristic $p > 0$.

We've already seen that F-split singularities need not be normal (although they are pretty close to normal since they are always weakly normal). The simplest example is $k[x, y]/(xy)$ which is F-split by Fedder's criterion by not normal.

F-split singularities need not be Cohen-Macaulay either. For example, $k[x, y, u, v]/((x, y) \cap (u, v))$ is F-split (this can be verified either using Fedder's criterion or the gluing methods we used for 1-dimensional varieties). Of course, this example is not normal and so one might hope for an example of a F-split normal singularity that is not Cohen-Macaulay. We provide one here.

We'll now look at the characteristic $p > 0$ situation, but first we need to have a brief discussion about reflexive rank one sheaves.

Suppose that X is normal and integral and that D is an integral Weil divisor on X. Then $\mathcal{O}_X(D) = \{f \in K(X) | \operatorname{div}(f) + D \geq 0\}$. This sheaf is rank one (clearly) and reflexive. Reflexive in this case means one of the following equivalent definitions. A sheaf $\mathscr{M}$ on X is reflexive if:

- $\mathscr{M}^{**} := \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{M}, \mathcal{O}_X), \mathcal{O}_X) \cong \mathscr{M}$ via the natural map. Equivalently...
- $\mathscr{M}$ is torsion free and for any open set $i : U \subseteq X$ such that $X \setminus U$ has codimension 2 (or more), $i_* \mathscr{M}|_U = \mathscr{M}$.

Remark 1.6. The second condition allows us to treat Weil divisors like Cartier divisors by setting $U = \operatorname{reg}(X)$. Generally speaking, $\mathcal{O}_X(D) \otimes \mathcal{O}_X(F) \neq \mathcal{O}_X(D+F)$ but up to double-dual $(\_^{**})$, it is true. Furthermore, for any such U, the operation i_* induces an equivalence of categories between reflexive sheaves on U and reflexive sheaves on X.

Explicitly, a map of reflexive sheaves is an isomorphism if and only if it is an isomorphism in codimension 1.

Lemma 1.7. *If X is as above and F-finite, then a torsion-free sheaf $\mathscr{M}$ is reflexive if and only if $F^e_* \mathscr{M}$ is $\mathcal{O}_X$-reflexive.*

Proof. Choose $U \subseteq X$ such that $X \setminus U$ has codimension at least 2, U is regular, and also such that $\mathscr{M}|_U$ is locally free (X is normal, so $\mathscr{M}$ is already locally free at every codimension 1 point, whose stalks are PIDs). This also implies that $F^e_* \mathscr{M}|_U$ is also locally free as an $\mathcal{O}_U$-module since $F^e_* \mathcal{O}_U$ is a locally free $\mathcal{O}_U$-module. Now, $F^e_* \mathscr{M}$ is $\mathcal{O}_X$-reflexive if and only if $i_*(F^e_* \mathscr{M}|_U) \cong F^e_* \mathscr{M}$. But that is clearly equivalent to $i_*(\mathscr{M}|_U) \cong \mathscr{M}$ which is the same thing as saying that $\mathscr{M}$ is reflexive. $\square$

We now turn to the question of whether F-split singularities deform. We consider the following situation. Suppose that R is a local ring and $f \in R$ is a regular element. If R/f is F-split, when can we conclude that R is F-split? The easiest approach would be to show that every map $\phi : F^e_*(R/f) \to R/f$ extends to a map $\bar{\phi} : F^e_* R \to R$. So we have to ask ourselves whether this is the case. We will show it is the case when R is Gorenstein, and show it is not the case when R is not Gorenstein (even if R is Cohen-Macaulay and normal).

Lemma 1.8 is also closely related to the fact that the set of Frobenius actions on $H^d_{\mathfrak{m}}(R)$ is generated by the natural Frobenius action $F^e : H^{\dim R}_{\mathfrak{m}}(R) \to H^{\dim R}_{\mathfrak{m}}(R)$; see [LS01].

Lemma 1.8. *Suppose that R is an F-finite Gorenstein local ring. By dualizing the natural map $G : R \to F^e_* R$ (apply $\operatorname{Hom}_R(\_, \omega_R)$), we construct the map*

$$\Psi : F^e_* \omega_R \to \omega_R$$

By fixing any isomorphism of ω_R with R (which we can do since R is Gorenstein), we obtain a map which we also call Ψ,

$$\Psi : F^e_* R \to R.$$

This map Ψ is an $F^e_* R$-module generator of $\operatorname{Hom}_R(F^e_* R, R)$.

Proof. First note that the choices we made in the setup of the lemma are all unique up to multiplication by a unit. Therefore, these choices are irrelevant in terms of proving that Ψ is an $F^e_* R$-module generator. Suppose that ϕ is an arbitrary $F^e_* R$-module generator of $\operatorname{Hom}_R(F^e_* R, R)$, and so we can write $\Psi(\_) = \phi(d \cdot \_)$ for some $d \in F^e_* R$. Using the same isomorphisms we selected before, we can view ϕ as a map $F^e_* \omega_R \to \omega_R$. By duality for a finite morphism, we obtain $\phi^\vee : R \to F^e_* R$. Note now that $G(\_\_) = d \cdot \phi^\vee(\_\_)$. But G sends 1 to 1 which implies that d is a unit and completes the proof. $\square$

Before continuing we need the following observation.

Lemma 1.9. *Given an effective Weil divisor D in a normal affine scheme $X = \operatorname{Spec} R$, the maps $\phi : \operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ which are compatible with D exactly coincide with the image of*

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e - 1)D), \mathcal{O}_X) \to \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X).$$

Proof. Suppose we have a $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ compatible with D. In other words, $\phi(F^e_* \mathcal{O}_X(-D)) \subseteq \mathcal{O}_X(-D)$. Twisting by $\mathcal{O}_X(D)$. $\square$

We now prove our desired extension result. A similar argument (involving local duality) was used in the characteristic $p > 0$ inversion of adjunction result of [HW02, Theorem 4.9].

Proposition 1.10. *Suppose that X is normal and $D \subseteq X$ is an effective Weil divisor which is also normal. Further suppose that D is Cartier in codimension 2 and that $(p^e - 1)(K_X + D)$ is Cartier. Then the natural map of $F^e_* \mathcal{O}_X$-modules:*

$$\Phi : \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e - 1)D), \mathcal{O}_X) \to \mathscr{H}\mathrm{om}_{\mathcal{O}_D}(F^e_* \mathcal{O}_D, \mathcal{O}_D).$$

induced by restriction is surjective.

Proof. The statement is local so we may assume that $X = \operatorname{Spec} R$ where R is the spectrum of a local ring. The module $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X((p^e - 1)D), \mathcal{O}_X) \cong F^e_* \mathcal{O}_X((1 - p^e)(K_X + D))$ which is isomorphic to $F^e_* \mathcal{O}_X = F^e_* R$ because we restricted to the local setting.

Thus the image of Φ is cyclic as an $F^e_* \mathcal{O}_D$-module which implies that the image of Φ is a reflexive $F^e_* \mathcal{O}_D$-module. Therefore, it is sufficient to prove that Φ is surjective at the codimension one points of D (which correspond to codimension two points of X). We now assume that $X = \operatorname{Spec} R$ is the spectrum of a two dimensional normal local ring and that D is a Cartier divisor defined by a local equation $(f = 0)$. Since D is normal and one dimensional, D is Gorenstein, and so X is also Gorenstein.

Consider the following diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\times f"] \arrow[d, "1 \mapsto f^{p^e - 1}"'] & R \arrow[r] \arrow[d, "1 \mapsto 1"] & R/f \arrow[r] \arrow[d, "1 \mapsto 1"] & 0 \\
0 \arrow[r] & F^e_* R \arrow[r, "F^e_* \times f"] & F^e_* R \arrow[r] & F^e_*(R/f) \arrow[r] & 0.
\end{tikzcd}$$

Apply the functor $\operatorname{Hom}_R(\_\_, \omega_R)$ and note that we obtain the following diagram of short exact sequences.

$$\begin{tikzcd}
0 \arrow[r] & \omega_R \arrow[r, "\times f"] & \omega_R \arrow[r] & \omega_{R/f} \cong \operatorname{Ext}^1_R(R/f, \omega_R) \arrow[r] & 0 \\
0 \arrow[r] & F^e_* \omega_R \arrow[r, "F^e_* \times f"] \arrow[u, "\alpha"] & F^e_* \omega_R \arrow[r] \arrow[u, "\beta"] & F^e_* \omega_{R/f} \cong \operatorname{Ext}^1_R(F^e_*(R/f), \omega_R) \arrow[r] \arrow[u, "\delta"] & 0
\end{tikzcd}$$

The sequences are exact on the right because R is Gorenstein and hence Cohen-Macaulay. Note that by Lemma 1.8, we see that δ and α can be viewed as $F^e_* R$-module generators of the modules $\operatorname{Hom}_{R/f}(F^e_*(R/f), R/f) \cong \operatorname{Hom}_{R/f}(F^e_* \omega_{R/f}, \omega_{R/f})$ and $\operatorname{Hom}_R(F^e_* R, R) \cong$

$\mathrm{Hom}_R(F^e_* \omega_R, \omega_R)$ respectively. Furthermore, the map labeled β can be identified with $\alpha \circ \big(F^e_*(\times f^{p^e-1})\big)$.

But the diagram proves exactly that the map $\beta \in \mathrm{Hom}_R(F^e_* R, R)$ restricts to a generator of $\mathrm{Hom}_{R/f}(F^e_* \omega_{R/f}, \omega_{R/f})$ which is exactly what we wanted to prove. $\square$

Remark 1.11. If D is not assumed to be normal but instead assumed to be S2 and Gorenstein in codimension 1, the proof goes through without change.

Corollary 1.12. *If R is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p > 0$ and R/f is normal and F-split, then R is also F-split.*

What happens if we relax these normal and $\mathbb{Q}$-Gorenstein conditions?

Example 1.13. [Fed83] [Sin99] Consider $R = k[u, v, y, z]/(uv, uz, z(v - y^2))$. Note that $(uv, uz, z(v - y^2)) = ((u, z) \cap (v, z)) \cap (u, v - y^2)$ and so R is not normal. We will show it is not F-pure but that there is a hypersurface through the origin that is F-pure.

First, if it was F-pure, then there would be a splitting $\phi : F^e_* R \to R$ which would induce a splitting of $k[u, v, y, z]$ (by Fedder's Lemma) and also be compatible with the minimal primes of R, (u, z), (v, z) and also $(u, v - y^2)$. All of those rings are F-split, and so that isn't a problem. However, $(v, z) + (u, v - y^2) = (u, v, y^2, z)$ isn't reduced so this is impossible.

Now, consider $R/y = k[u, v, z]/(uv, uz, zv)$ which is F-pure.

Of course, you may view this as cheating since R is not normal (although it is still Cohen-Macaulay). One can construct normal examples as well, see [Sin99, Theorem 1.1]. We're going to abandon F-splitting for a little while now, and we'll consider the following condition.

Lemma 1.14. *If R is an F-split local ring, then the natural map $\Psi : F_* \omega_R \to \omega_R$ of Lemma 1.8 is surjective. Furthermore, if R is quasi-Gorenstein, then the converse also holds.*

Proof. If R is F-split, we have a composition which is an isomorphism $R \to F_* R \to R$. Dualizing this gives us

$$\omega_R \xleftarrow{\Psi} F_* \omega_R \longleftarrow \omega_R$$

which is also an isomorphism. Thus Ψ is surjective.

Conversely, if Ψ is surjective and R is quasi-Gorenstein, then $\omega_R \cong R$ and we have a surjective map $\Psi : F_* R \to R$. $\square$

Definition 1.15. [Fed83] A Cohen-Macaulay ring R is called F-injective if the natural map $\Psi : \omega_R \to \omega_R$ is surjective.

Remark 1.16. You might ask why he called this condition F-injective and not F-surjective? It is because Ψ is the local dual of the Frobenius map $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$ on local cohomology and Ψ is surjective if and only if that map is injective. More generally, in the non-Cohen-Macaulay case, he said that R was F-injective if $h^i(F_* \omega_R^{\bullet}) \to h^i(\omega_R^{\bullet})$ is surjective for every i.

Furthermore, we have the following.

Proposition 1.17. [Fed83] *Suppose that R is Cohen-Macaulay and R/f is F-injective. Then R is F-injective.*

Proof. Consider the following diagram of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\times f"] \arrow[d, "1 \mapsto f^{p^e-1}"'] & R \arrow[r] \arrow[d, "1\mapsto 1"] & R/f \arrow[r] \arrow[d, "1\mapsto 1"] & 0 \\
0 \arrow[r] & F^e_* R \arrow[r, "F^e_* \times f"] & F^e_* R \arrow[r] & F^e_*(R/f) \arrow[r] & 0.
\end{tikzcd}$$

Apply the functor $\mathrm{Hom}_R(\_\_, \omega_R)$ and note that we obtain the following diagram of short exact sequences.

$$\begin{tikzcd}
 & C \arrow[r, "\eta"] & D \arrow[r] & 0 & \\
0 \arrow[r] & \omega_R \arrow[r, "\times f"] \arrow[u] & \omega_R \arrow[r] \arrow[u] & \omega_{R/f} \cong \mathrm{Ext}^1_R(R/f, \omega_R) \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & F^e_*\omega_R \arrow[r, "F^e_* \times f"] \arrow[u, "\alpha"] & F^e_*\omega_R \arrow[r] \arrow[u, "\beta"] & F^e_*\omega_{R/f} \cong \mathrm{Ext}^1_R(F^e_*(R/f), \omega_R) \arrow[r] \arrow[u, "\delta"] & 0
\end{tikzcd}$$

where C and D are the cokernels of α and β respectively. Thus, $C = \omega_R/\Psi_R(F^e_*\omega_R)$ and $D = \omega_R/\Psi_R(F^e_* f^{p^e-1}\omega_R))$. We have a natural surjective map

$$\mu : D = \omega_R/\Psi(F^e_* f^{p^e-1}\omega_R)) \to \omega_R/\Psi(F^e_*\omega_R)) = C$$

and we see that $\mu \circ \eta : C \to C$ is simply multiplication by f. But η surjects and thus so does $\mu \circ \eta$. But this contradicts Nakayama's lemma. $\square$

Because of this, Fedder suggested that normal, Cohen-Macaulay and F-injective might be a closer match to rational singularities than F-purity. There was some evidence for this. In particular, Fedder showed that certain classes of hypersurfaces (defined over $\mathbb{Z}$) had rational singularities over $\mathbb{C}$ if and only if for all sufficiently large $p > 0$, the singularity viewed modulo p had F-pure (equivalently, F-injective) singularities. Notice that this doesn't allow $x^3+y^3+z^3$ because that does not have F-pure singularities for $p = 2 \bmod 3$. Elkies has since shown that for cones over planar elliptic curves (none of which have rational singularities), they are supersingular (and thus ordinary) for infinitely many p.

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