F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/2-2010

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1. Criteria for local Frobenius splitting I (Fedder's criteria)

Now we need some notation.

Definition 1.1. Suppose that S is a ring and I is an ideal. If $\psi : F^e_* S \to S$ is an S-linear map, we say that I is *ϕ-compatible* if $\psi(F^e_* I) \subseteq I$.

Remark 1.2. Clearly if I is ψ-compatible, then ψ induces a map on R/I.

Remark 1.3. Remember that for ideals I, J, the notation $I : J$ is all the elements $r \in R$ such that $rJ \subseteq I$. In other words, it is the same as $Ann_R(J + I/I)$.

Theorem 1.4. [Fed83]*[Fedder's Lemma] Suppose that S is a regular local ring and that $R = S/I$. The set of $\phi \in \mathrm{Hom}_S(F^e_* S, S)$ which satisfy $\phi(F^e_* I) \subseteq I$ is equal to $F^e_*(I^{[p^e]} : I) \cdot \mathrm{Hom}_S(F^e_* S, S) \cong F^e_*(I^{[p^e]} : I)$ and those which induce the zero map on $R = S/I$ correspond to $I^{[p^e]}$. In conclusion, $\mathrm{Hom}_R(F^e_* R, R) \cong F^e_*(I^{[p^e]} : I)/(I^{[p^e]})$.*

Proof. Let $\Phi \in \mathrm{Hom}_S(F^e_* S, S)$ be a generating map. We will first show the following lemma.

Lemma 1.5. *For any ideals $I, J \subseteq S$, we have $\Phi(F^e_* J) \subseteq I$ if and only if $I^{[p^e]} \supseteq J$.*

Proof. The ($\Leftarrow$) direction is easier and we start with that. We claim that $\phi(F^e_* I^{[p^e]}) \subseteq I$. To see this, note that if $I = (x_1, \dots, x_n)$, then $I^{[p^e]} = (x_1^{p^e}, \dots, x_n^{p^e})$ and so if $z \in I^{[p^e]}$, then $z = \sum a_i x_i^{p^e}$. Then $\Phi(z) = \Phi(\sum a_i x_i^{p^e}) = \sum x_i \phi(a_i)$. The first direction then immediately follows.

Conversely, suppose that $\Phi(F^e_* I) \subseteq J$. We choose $y_1, \dots, y_m$ to be a basis for $F^e_* S$ over S (we can obviously project on to each factor via multiplication of Φ by elements of $F^e_* S$, and any map $\phi : F^e_* S \to S$ is a sum of such projections). So, we need $F^e_* I \subseteq \oplus J \cdot y_i = J \cdot F^e_* S = F^e_* J^{[p^e]}$. In other words, $I \subseteq J^{[p^e]}$ as desired. □

I claim that a map $\phi : F^e_* S \to S$ sends $F^e_* I$ into I if and only if $\phi \in F^e_*(I^{[p^e]} : I) \cdot \Phi$. To see this, write $\phi = z \cdot \Phi$ for some $z \in F^e_* S = S$. Then $\phi(F^e_* I) \subseteq I$ if and only if $\Phi(F^e_* zI) \subseteq I$ which happens if and only if $zI \subseteq I^{[p^e]}$, in other words, if and only if $z \in I^{[p^e} : I$. Thus $\phi \in F^e_*(I^{[p^e]} : I) \cdot \Phi$ if and only if $\phi(F^e_* I) \subseteq I$.

For the second statement, suppose that $\phi \in I^{[p^e]} \cdot \Phi$. Thus for every $x \in F^e_* S$, $\phi(x) \in I$ (use the previous lemma with $J = I^{[p^e]}$ $I = I$). Thus the induced map on $R = S/I$ is the zero map. Conversely, suppose that $\phi \in F^e_*(I^{[p^e]} : I) \cdot \Phi$ but $\phi \notin I^{[p^e]} \cdot \Phi$. Thus there is some $x \in F^e_* S$ such that $\phi(x) \notin I$ and so the induced map on $R = S/I$ is non-zero. □

Corollary 1.6 (Fedder's criteria)**.** *If $(S, \mathfrak{m})$ is a F-finite regular local ring and $R = S/I$, then R is F-split if and only if $I^{[p^e]} : I$ is not contained in $\mathfrak{m}^{[p^e]}$.*

Proof. For $\bar{\phi} \in \operatorname{Hom}_R(F^e_* R, R)$ (induced from $\phi : F^e_* S \to S$) to be surjective, it must contain 1 in it's image. This happens if and only if $\phi \notin \mathfrak{m}^{[p^e]} \cdot \Phi$ (where Φ is in the previous proof). Such a map exists if and only if $I^{[p^e]} : I \not\subseteq \mathfrak{m}^{[p^e]}$. □

Remark 1.7. If $I = (f)$ is a principal ideal, then $I^{[p^e]} : I = (f^{p^e-1})$ which is very easy to compute by hand. In many cases, the colon's can be done via a computer.

We now do several examples.

Example 1.8. The following rings are F-split.

(1) $R = k[x_1, \dots, x_n]/(x_1 \cdot \dots \cdot x_n)$. Notice that $(x_1 \cdot \dots \cdot x_n)^{p^e-1} \not\subseteq (x_1^{p^e}, \dots, x_n^{p^e}) = m^{[p^e]}$.
(2) $R = k[x, y, z]/(x^2 - yz)$. Notice that $(x^2 - yz)^{p^e-1}$ has a term $(yz)^{p^e-1}$ which does not appear in $m^{[p^e]}$.
(3) $R = k[x, y, z]/(x^2 - y^2 z)$ if the characteristic of k is not 2. In this case, $(x^2 - y^2 z)^{p-1}$ has a term $\binom{p-1}{(p-1)/2} x^{p^e-1} y^{p^e-1} z^{\frac{p^e-1}{2}}$ and so the question is whether p divides the binomial coefficient. But it is clear that it does not.
(4) $R = k[x, y, z]/(x^3 + y^3 + z^3)$ if the characteristic of k is 7 (check it yourself). One can also check that it is not F-split for characteristics $2, 3, 5$ and more generally if $p = 2$ mod 3.

Fedder's Lemma suggests the following question.

Question 1.9. Given an arbitrary ring T with quotient $R = T/I$. Is it true that every map $\phi \in \operatorname{Hom}_R(F^e_* R, R)$ is induced from a map $\phi \in \operatorname{Hom}_T(F^e_* T, T)$?

The answer to this question is no as the following example demonstrates:

Example 1.10. Consider $S = k[x, y, z]$, $T = k[x, y, z]/(x^2 - yz)$ and $R = k[x, y, z]/(x, y)$. The map $\Phi_R : F_* R \to R$ which sends z^{p-1} to 1 and the other z^i to zero is induced by maps written as $\Phi_S(w \cdot \_\_)$ where Φ_S is the $F_* S$-module generator of $\operatorname{Hom}_S(F_* S, S)$ discussed above and w is an element of the coset $(xy)^{p-1} + (x^p, y^p)$. We have to ask ourselves whether such a w can be inside $((x^2 - yz)^{p-1}) + (x^p, y^p)$, and the answer is clearly no.

2. Very basic facts about Frobenius splitting

First we discuss the difference between F-purity and F-splitting.

Definition 2.1. A ring R of characteristic $p > 0$ is said to be *F-pure* if for every R-module M, the map $M \otimes R \to M \otimes F_* R$ is pure.

Clearly an F-split ring is F-pure. Furthermore, if R is F-finite, then an F-pure ring is also F-split (see The notion of F-purity is much better behaved outside the F-finite context. However, we won't be going there.

In an F-finite scheme, F-purity is used interchangably with local F-splitting. An F-splitting (without a "local" qualifier) is always viewed as a global statement.

Here we list (and prove) a number of basic facts about Frobenius splittings, again mostly in the local context.

Theorem 2.2. *Suppose that R is an F-finite ring. Then the following hold:*

(a) *If R is Frobenius split (F-split) then R is reduced.*

(b) *If R_Q is Frobenius split for some $Q \in \operatorname{Spec} R$, then R is Frobenius split in a neighborhood of Q.*
(c) *R is F-split if and only if $R_{\mathfrak{m}}$ is F-split for every maximal ideal $\mathfrak{m}$ if and only if R_Q is F-split for every prime ideal Q.*
(d) *If $R \subseteq S$ is a split inclusion of rings and S is F-split, then R is also F-split.*
(e) *If R is F-split, then for every minimal prime $\mathfrak{q} \subseteq R$, $R/\mathfrak{q}$ is also F-split.*
(f) *If $\phi : F^e_* R \to R$ is any R-linear map and I and J are ϕ-compatible ideals, then so is $I + J$, $I \cap J$, $\sqrt{I}$, and also $I : \mathfrak{a}$ for any ideal $\mathfrak{a}$.*

3. (Weak/Semi)normality and Frobenius splitting

Today we'll prove that a F-split ring is weakly normal and thus seminormal (so first I'll define these terms).

First we'll talk about some hand-wavy geometry. Seminormality (and weak normality) are ways of forcing all gluing of your scheme is as transverse as possible. So first what is "gluing"?

Suppose that R is an F-finite reduced ring with normalization R^N (domain of finite type over a field is fine). The semi-normalization R^{SN} (and weak normalization R^{WN} of R is a partial normalization of R inside R^N). Since R is F-finite it is excellent, and so all these extensions are finite extensions (ie, we don't have to worry about extreme funny-ness).

Definition 3.1. [AB69], [GT80], [Swa80] A finite integral extension of reduced rings $i : A \subset B$ is said to be *subintegral* (respectively *weakly subintegral*) if

(i) it induces a bijection on the prime spectra, and
(ii) for every prime $P \in \operatorname{Spec} B$, the induced map on the residue fields, $k(i^{-1}(P)) \to k(P)$, is an isomorphism (respectively, is a purely inseparable extension of fields).

Remark 3.2. A subintegral extension of rings has also been called a quasi-isomorphism; see for example [GT80].

Remark 3.3. Condition (ii) is unnecessary in the case of extensions of rings of finite type over an algebraically closed field of characteristic zero.

Definition 3.4. [GT80, 1.2], [Swa80, 2.2] Let $A \subset B$ be a finite extension of reduced rings. Define ${}^+_B A$ to be the (unique) largest subextension of A in B such that $A \subset {}^+_B A$ is subintegral. This is called the *seminormalization of A inside B*. A is said to be *seminormal in B* if $A = {}^+_B A$. If A is seminormal inside its normalization, then A is called *seminormal*.

Definition 3.5. [AB69], [Yan85], [RRS96, 1.1] Let $A \subset B$ be a finite extension of reduced rings. Define ${}^*_B A$ to be the (unique) largest subextension of A in B such that $A \subset {}^*_B A$ is weakly subintegral. This is called the *weak normalization of A inside B*. A is said to be *weakly normal in B* if $A = {}^*_B A$. If A is weakly normal inside its normalization, then A is called *weakly normal*.

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