F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/17-2010

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We also state Grothendieck duality.

Theorem 0.1. [Har66] *Given a map of schemes $f : Y \to X$ of finite type, there exists a functor $f^! : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$. If furthermore, f is proper then one has the following:*

(i) $R\mathscr{H}\mathrm{om}^{\bullet}_{\mathcal{O}_X}(Rf_*\mathscr{F}^{\bullet}, \mathscr{G}^{\bullet}) \cong Rf_* R\mathscr{H}\mathrm{om}^{\bullet}_{\mathcal{O}_Y}(\mathscr{F}^{\bullet}, f^!\mathscr{G}^{\bullet})$ *where* $\mathscr{F}^{\bullet}, \mathscr{G}^{\bullet} \in D^b_{\mathrm{coh}}(X)$.

(ii) $f^!\omega^{\bullet}_X$ *is a dualizing complex for Y (denoted now by $\omega^{\bullet}_Y$).*

(iii) *If $f : Y \to X$ is a finite map (for example, a closed immersion), $f^!$ is identified with $R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \_\_)$ (viewed then as a module on Y).*

We will also use Kodaira vanishing and a relative version, Grauert-Riemenschneider vanishing.

Theorem 0.2 (Kodaira Vanishing)**.** *Suppose that X is a smooth variety of characteristic zero and $\mathscr{L}$ is an ample line bundle on X. Then $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for $i > 0$ or dually, $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < \dim X$.*

Theorem 0.3. [GR70] *Suppose that $\pi : \widetilde{X} \to X$ is a proper map of algebraic varieties in characteristic zero with $\widetilde{X}$ smooth. Then $R^i\pi_*\omega_{\widetilde{X}} = 0$ for $i > 0$.*

Remark 0.4. Both of these theorems *FAIL* in characteristic $p > 0$.

0.1. **The Cohen-Macaulay and Gorenstein conditions for section rings.** To illustrate these previous notions, let us consider section rings of projective varieties with respect to ample divisors. Throughout this section, X will denote a smooth[1] projective variety over an algebraically closed field of characteristic 0 also with canonical divisor K_X. Let A be a (very (very)) ample divisor on X (ample is actually fine, but it is harmless to make it more ample for the purposes of the examples in this section).

Let $S = \oplus H^0(X, \mathcal{O}_X(nA))$ denote the section ring of S with respect to A and suppose that $\mathfrak{m} = S_+$ is the irrelevant ideal. If $Y = \operatorname{Spec} S$, then $U = \operatorname{Spec} S \setminus V(\mathfrak{m})$ is a k^*-bundle over $q : U \to X$ (far from the trivial bundle though). If S is generated in degree one, this is an easy exercise, for the more general case see for example [HS04]. We use $i : U \to Y$ to denote the inclusion.

Thus given any divisor D on X, $\oplus H^0(X, \mathcal{O}_X(D + nA))$ is the sheaf corresponding to a divisor on Y. In fact, it corresponds to the divisor q^*D extended in the unique way over the irrelevant point of Y (in other words, it corresponds to i_*q^*D). We use D_Y to denote this corresponding divisor on Y and make the easy observation that $n(D_Y) = (nD)_Y$. What's more important, is that $\oplus H^0(X, \mathcal{O}_X(K_X + nA))$ *IS* the canonical module ω_Y of Y (this basically follows from what we've described since q is just a k^*-bundle).

[1]Large parts of the section also work if X is normal, and all the results of the section hold if one assumes that X has rational singularities.

Let us first consider what this means for the quasi-Gorenstein and $\mathbb{Q}$-Gorenstein conditions. Since $\omega_Y = \mathcal{O}_Y(lK_Y)$ is a graded S-module, it will be free if and only if $\mathcal{O}_Y(lK_Y)$ is a locally free graded module (which means if and only if $\mathcal{O}_Y(nK_Y)$ is a line bundle). The graded line bundles on S are just S with a shift. In summary

Lemma 0.5. *S is quasi-Gorenstein if and only if $K_X \sim nA$ for some integer n (possibly equal to zero). Furthermore, S is $\mathbb{Q}$-Gorenstein if and only if $mK_X \sim nA$ for some integers n, m not both zero.*

Proof. We first prove the second statement which is slightly harder than the first statement. If S is $\mathbb{Q}$-Gorenstein, then $\oplus_k H^0(X, \mathcal{O}_X(mK_X + kA))$ is isomorphic to $S(n)$ for some integer n. But $\mathcal{O}_X(mK_X)$ is completely determined as an $\mathcal{O}_X$-module by $\oplus_k H^0(X, \mathcal{O}_X(mK_X + kA))$ and if it is isomorphic to $S(n) = \oplus_k H^0(X, \mathcal{O}_X(nA + kA))$, then $nA \sim mK_X$ as desired. The converse simply reverses this. □

Corollary 0.6. *If X is such that $K_X \sim 0$, then for any section ring S, S is quasi-Gorenstein.*

Remark 0.7. We also see that it is possible that for some A the section ring is $\mathbb{Q}$-Gorenstein, while for other A the section ring of the same variety is not $\mathbb{Q}$-Gorenstein. Furthermore, there are varieties with no section ring (with respect to an ample divisor) being $\mathbb{Q}$-Gorenstein.

Using something called local duality, the Cohen-Macaulay condition can also be translated as follows (even for non-graded local rings).

Lemma 0.8. *Suppose that $(S, \mathfrak{m})$ is a local ring. Then*

- *S is* Cohen-Macaulay *if and only if $H^i_{\mathfrak{m}}(S) = 0$ for $i < \dim S$.*

Remark 0.9. Literally, local duality says that the complex $R\Gamma_{\mathfrak{m}}(S)$ is dual to the complex $\omega^{\bullet}_S$.

If we are working with a normal section ring as before, then $H^0_{\mathfrak{m}}(S) = H^1_{\mathfrak{m}}(S) = 0$ (the first follows from the fact that S is reduced, the second from the fact that S is normal, see for example [Har77, Chapter III, Exercise 3.4]). Therefore, to show the Cohen-Macaulay condition, we only need to show the vanishing of the higher $H^i_{\mathfrak{m}}(S)$ for $1 < i \leq \dim S - 1 = \dim X$. As noted before, $(H^i_{\mathfrak{m}}(S))_n = H^{i-1}(X, \mathcal{O}_X(n))$ and so we have the following:

Lemma 0.10. *A section ring S of a projective variety X is Cohen-Macaulay if and only if $H^j(X, \mathcal{O}_X(n)) = 0$ for $0 < j < \dim X$ and all $n \geq 0$.*

Proof. The Cohen-Macaulay condition certainly implies the vanishing by the discussion above. Furthermore $H^j(X, \mathcal{O}_X(n)) = 0$ for $n < 0$ by Kodaira-vanishing (at least if X is smooth although Kodaira vanishing also holds for rational singularities) which proves the converse. □

One should thus note that it is possible that some section rings of a projective variety can fail to be Cohen-Macaulay, while others are Cohen-Macaulay (take a very high Veronese embedding). In particular, X has a section ring that is Cohen-Macaulay if and only if $H^j(X, \mathcal{O}_X) = 0$ for all $0 < j < \dim X$.

Watanabe's definition of rational singularities also can be restated as follows. Recall that he said that S has rational singularities if and only if S is Cohen-Macaulay and $a(S) < 0$ where $a(S) := \max\{n | (H^{\dim S}_{\mathfrak{m}}(S))_n \neq 0\}$.

Lemma 0.11. *A section ring S of a projective variety X has rational singularities if and only if $H^j(X, \mathcal{O}_X(n)) = 0$ for $0 < j \leq \dim X$ and all $n \geq 0$.*

Again, it is possible for some section rings to have rational singularities while other section rings do not have rational singularities.

We conclude with an example of a ring that is quasi-Gorenstein but not Cohen-Macaulay.

Example 0.12. Suppose that X is an Abelian surface (for example, the product of two elliptic curves). The irregularity of X is defined to be $\dim H^1(X, \mathcal{O}_X)$ and it an exercise in Hartshorne ([Har77, Chapter II, Section 8, Exercise 8.3(c)]) which shows that the irregularity is 2 (and in particular, non-zero).

0.2. **A definition of rational singularities.** Now we define rational singularities as well as resolutions of singularities.

Definition 0.13. Let X be a reduced scheme of (essentially) finite type over a field. We say that a map $\pi : \widetilde{X} \to X$ is a *resolution of singularities* if the following conditions are satisfied:

- (1) $\widetilde{X}$ is [regular / smooth], these notions agree in characteristic zero.
- (2) π is proper.
- (3) π is birational.

Remark 0.14. Resolutions of singularities exist in characteristic zero, [Hir64], [BM97], [BEV05], [Wło05], [Kol07]. Furthermore, there always exists a resolution satisfying the following properties.

- (a) π is projective, in other words, it is the blow-up of some (horrible) ideal.
- (b) π is an isomorphism on the locus where X is regular.
- (c) π is obtained by a sequence of blow-ups at smooth subvarieties (if $X \subseteq Y$ and Y is smooth, one may instead require that π is obtained by a sequence of blow-ups at smooth points of Y).
- (d) The reduced exceptional locus of π is a divisor with simple normal crossings (it looks analytically like $k[x_1, \ldots, x_n]/(\text{some product of the } x_i)$).

Now we define rational singularities.

Definition 0.15. A reduced local ring $(R, \mathfrak{m})$ of characteristic zero is said to have rational singularities if, for a given (equivalently any) resolution of singularities $\pi : \widetilde{X} \to X$, we have the following two conditions.

(i) $\pi_* \mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ (in other words, X is normal)
(ii) $R^i \pi_* \mathcal{O}_{\widetilde{X}} = 0$ for $i > 0$.

Proposition 0.16. *If X has rational singularities and $\pi : \widetilde{X} \to X$ is a resolution of singularities, then for any line bundle (or vector bundle) $\mathscr{L}$ on X, we have $H^i(X, \mathscr{L}) = H^i(\widetilde{X}, \pi^* \mathscr{L})$. In other words, cohomology of line bundles can be computed on a resolution.*

Proof. By the projection formula, $R^j \pi^* \mathscr{L} = 0$ for all $j > 0$. The statement then follows from the E_2 degeneration of the associated spectral sequence. $\square$

1. Other characterizations of rational singularities

Reinterpreting the rational singularities condition in the derived category gives us the following.

Definition 1.1. If X is a singular variety and $\pi : \widetilde{X} \to X$ is a resolution, then X has *rational singularities* if and only if $\mathcal{O}_X \to R\pi_* \mathcal{O}_{\widetilde{X}}$ is an isomorphism.

We will now apply Grothendieck duality to this definition. Consider the map $\mathcal{O}_X to R\pi_* \mathcal{O}_{\widetilde{X}}$. Apply the duality functor $R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\_\_, \omega_X^\bullet)$. This gives us a map

$$R\pi_* \omega_{\widetilde{X}}^\bullet \cong R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_{\widetilde{X}}, \pi^! \omega_X^\bullet) \cong R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(R\pi_* \mathcal{O}_{\widetilde{X}}, \omega_X^\bullet) \to R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^\bullet) \cong \omega_X^\bullet.$$

Now, because $\widetilde{X}$ is smooth, it is Gorenstein so $\omega_{\widetilde{X}}^\bullet = \omega_{\widetilde{X}}[\dim X]$. Grauert-Riemenschneider vanishing tells us then that $R^i \pi_* \omega_{\widetilde{X}}^\bullet = R^{i+d} \pi_* \omega_{\widetilde{X}} = 0$ for $i + d \neq 0$ or equivalently for $i \neq -d$. If X has rational singularities, we immediately see that $h^i \omega_X^\bullet = R^i \pi_* \omega_{\widetilde{X}}^{mydot} = 0$ for $i \neq -d$. Thus X is Cohen-Macaulay. Conversely, we also obtain the following characterization of rational singularities due to Kempf.

Lemma 1.2. [KKMSD73] *With the notation as above, X has rational singularities if and only if X is Cohen-Macaulay and $\pi_* \omega_X \cong \omega_X$.*

Remark 1.3. One always has an inclusion $\pi_* \omega_{\widetilde{X}} \subseteq \omega_X$ so in general, one only needs to check the surjectivity.

It is a standard exercise to show that $\pi_* \omega_X = \omega_X$ in a regular ring (all the coefficients in the relative canonical divisor are positive). Once you have this, you see that the definition of rational singularities is independent of the resolution.

We'll do a standard example of rational (and non-rational) singularities in the graded case, then we'll explore some consequences of Kempf's criterion for rational singularities.

Example 1.4. Consider the (graded) ring $R = k[x, y, z]/(x^n + y^n + z^n)$. We'll set $Y = \operatorname{Spec} k[x, y, z]$ with closed subscheme $X = \operatorname{Spec} R$. We notice that the singularities of X can be resolved by blowing-up the cone-point of X (maximal ideal of R) which is the origin of Y, yielding $\pi : \widetilde{Y} \to Y$ (with exceptional $\mathbb{P}^2 = E$) which restricts to $\pi : \widetilde{X} \to X$ (with exceptional curve C). Because X is a hypersurface it is Cohen-Macaulay, and so we need to show that $\pi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \cong \mathcal{O}_X(K_X)$. One can always assume that K_X and $K_{\widetilde{X}}$ agree where π is an isomorphism and furthermore, that $\mathcal{O}_X(K_X) \cong \mathcal{O}_X$ and $\mathcal{O}_Y(K_Y) = \mathcal{O}_Y$ since X is a hypersurface in $Y = \mathbb{A}^n$. Thus, we need to compute $K_{\widetilde{X}/X} = K_{\widetilde{X}} = K_{\pi|_{\widetilde{X}}}$ the relative canonical divisor of $\pi|_{\widetilde{X}}$. If this divisor is effective, then $\pi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) = \mathcal{O}_X$ (what sections of $\mathcal{O}_X$ have poles along a divisor at a point). If it's not effective, then $\pi_* \mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}) \subsetneq \mathcal{O}_X$ since now we are requiring sections to vanish to some order at the maximal ideal.

We know the relative canonical divisor of π though, it's simply $\mathcal{O}_{\widetilde{Y}}(2E)$ by [Har77, Chapter II, Exercise 8.5(b)]. By the adjunction formula, $(2E + \widetilde{X})|_{\widetilde{X}} = (K_Y + \widetilde{X})|_{\widetilde{X}} = K_{\widetilde{X}}$. On the other hand, we know that $(nE + \widetilde{X})|_{\widetilde{X}} = \pi^* X|_{\widetilde{X}} \sim 0$ on $\widetilde{X}$. Thus, $K_{\widetilde{X}} \sim (2 - n)C$ since $E|_{\widetilde{X}} = C$.

As an easy consequence, we see that X has rational singularities if and only if $n = 1, 2$ and otherwise does not have rational singularities. Recall that the same singularity had F-split singularities if and only if $n = 1 \mod 3$.

Remark 1.5. You might ask where the adjunction formula comes from? If you have a hypersurface H on a Cohen-Macaulay variety X (if X is normal, the same statement holds because one can restrict to the Cohen-Macaulay locus which agrees with X outside a set of codimension 3), then we have a short exact sequence

$$0 \to \mathcal{O}_X(-H) \to \mathcal{O}_X \to \mathcal{O}_H \to 0.$$

Applying $R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\_\_, \omega_X^\bullet)$ gives us

$$\omega_H^\bullet = R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{O}_H, \omega_X^\bullet) \to \omega_X^\bullet \to \omega_X^\bullet(H) \to ...$$

If we take cohomology, we get

$$0 \to \omega_X \to \omega_X(H) \to \omega_H \to h^{-\dim X+1}(\omega_X^\bullet) = 0$$

If X is also normal, this is exactly the statement $K_X|_H = K_H$.

First, we look at Boutot's theorem (remember, we already showed that a summand of an F-split ring is always F-split).

Theorem 1.6. [Bou87] *If $R \subseteq S$ is an extension of normal domains such that R is a direct sum of S, then if S has rational singularities, so does R.*

Proof. We first claim that there exists resolutions of singularities $\alpha : \widetilde{X} \to X = \operatorname{Spec} X$ and $\beta : \widetilde{Y} \to Y = \operatorname{Spec} S$ making a commutative diagram:

$$\begin{array}{ccc} \widetilde{X} & \xleftarrow{\ \gamma\ } & \widetilde{Y} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X & \xleftarrow[\ \delta\ ]{} & Y \end{array}$$

To see this, first resolve the singularities of X by a blow-up of an ideal, and then blow-up the extension of that ideal on Y (giving $Y' \to Y$, that will give you a digram) and then further resolve the singularities of Y'. If we write down the derived category version of this diagram, we get

$$\begin{array}{ccc} R\alpha_*\mathcal{O}_{\widetilde{X}} & \longrightarrow & R\beta_*\mathcal{O}_{\widetilde{Y}} \\ \uparrow & & \uparrow{\scriptstyle f} \\ \mathcal{O}_X & \longrightarrow & \mathcal{O}_Y \end{array}$$

This gives us the following composition:

$$\mathcal{O}_X \to R\alpha_*\mathcal{O}_{\widetilde{X}} \to R\beta_*\mathcal{O}_{\widetilde{Y}} \cong \mathcal{O}_Y \to \mathcal{O}_X$$

which is an isomorphism. Thus $\mathcal{O}_X \to R\alpha_*\mathcal{O}_{\widetilde{X}}$ splits (has a left/right inverse) in the derived category. One should note that $R\beta_*\mathcal{O}_{\widetilde{Y}}$ (or even S) is not necessarily in $D^b_{\mathrm{coh}}(X)$, simply because the map $R \subseteq S$ may not be finite / proper. They do live in $D^b(X)$ though. However, $D^b_{\mathrm{coh}}(X)$ is a full subcategory of $D^b(X)$ (see [Har66]) so we may still assume our splitting lives in $D^b_{\mathrm{coh}}(X)$.

Therefore, our result follows once we prove the following lemma.

Lemma 1.7. [Kov00] *With notation as above, if $\mathcal{O}_X \to R\alpha_*\mathcal{O}_{\widetilde{X}}$ splits in the derived category, then X has rational singularities.*

Proof. We will use Kempf's criterion for rational singularities. By assumption, we have a composition (which is an isomorphism)

$$\mathcal{O}_X \to R\alpha_* \mathcal{O}_{\widetilde{X}} \to \mathcal{O}_X$$

Applying $R\mathscr{H}om_{\mathcal{O}_X}(\_\_, \omega_X^{\bullet})$ we obtain the following composition (which is also an isomorphism in the derived category)

$$\begin{array}{ccccc} R\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \omega_X^{\bullet}) = \omega_X^{\bullet} & \longleftarrow & R\mathscr{H}om_{\mathcal{O}_X}(R\alpha_*\mathcal{O}_{\widetilde{X}}, \omega_X^{\bullet}) & \longleftarrow & \omega_X^{\bullet} \\ & & \| & & \\ & & R\alpha_* R\mathscr{H}om_{\mathcal{O}_{\widetilde{X}}}(R\alpha_*\mathcal{O}_{\widetilde{X}}, \omega_{\widetilde{X}}^{\bullet}) & & \\ & & \| & & \\ & & R\alpha_*\omega_{\widetilde{X}}^{\bullet} & & \\ & & \| & & \\ & & \alpha_*\omega_{\widetilde{X}}[\dim X] & & \end{array}$$

Thus $h^{-\dim X + i}\omega_X^{\bullet} = 0$ for $i \neq 0$, which implies that X is Cohen-Macaulay. On the other hand, taking cohomology at the $-\dim X$ place gives us

$$\omega_X \leftarrow \alpha_*\omega_{\widetilde{X}} \leftarrow \omega_X$$

where the left-most arrow is the natural inclusion (which is always injective). The fact that the composition is an isomorphism implies that the left-most arrow is also injective, and thus an isomorphism. □

□

References

[BM97] E. Bierstone and P. D. Milman: *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), no. 2, 207–302. MR1440306 (98e:14010)

[Bou87] J.-F. Boutot: *Singularités rationnelles et quotients par les groupes réductifs*, Invent. Math. **88** (1987), no. 1, 65–68. MR877006 (88a:14005)

[BEV05] A. M. Bravo, S. Encinas, and O. Villamayor: *A simplified proof of desingularization and applications*, Rev. Mat. Iberoamericana **21** (2005), no. 2, 349–458. MR2174912

[GR70] H. Grauert and O. Riemenschneider: *Verschwindungssätze für analytische Kohomologiegruppen auf komplexen Räumen*, Invent. Math. **11** (1970), 263–292. MR0302938 (46 #2081)

[Har66] R. Hartshorne: *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR0222093 (36 #5145)

[Har77] R. Hartshorne: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[Hir64] H. Hironaka: *Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II*, Ann. of Math. (2) 79 (1964), 109–203; ibid. (2) **79** (1964), 205–326. MR0199184 (33 #7333)

[HS04] E. Hyry and K. E. Smith: *Core versus graded core, and global sections of line bundles*, Trans. Amer. Math. Soc. **356** (2004), no. 8, 3143–3166 (electronic). MR2052944 (2005g:13007)

[KKMSD73] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat: *Toroidal embeddings. I*, Lecture Notes in Mathematics, Vol. 339, Springer-Verlag, Berlin, 1973. MR0335518 (49 #299)

[Kol07] J. Kollár: *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, Princeton, NJ, 2007. 2289519 (2008f:14026)

[Kov00] S. J. Kovács: *A characterization of rational singularities*, Duke Math. J. **102** (2000), no. 2, 187–191. MR1749436 (2002b:14005)

[Wło05] J. Włodarczyk: *Simple Hironaka resolution in characteristic zero*, J. Amer. Math. Soc. **18** (2005), no. 4, 779–822 (electronic). MR2163383