F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/14-2010

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As promised, we will also attempt to describe $F_*\mathcal{O}_{\mathbb{P}^n}$.

Example 0.1. Suppose that $X = \mathbb{P}^n$. We first identify the possibly summands that can appear (as Christopher Hacon pointed out in class today), write $F_*\mathcal{O}_{\mathbb{P}^n} = \mathcal{O}_X \oplus \mathcal{O}_X(a_2) \oplus \cdots \oplus \mathcal{O}_X(a_{p^n})$ where the $a_i < 0$ are integers. Note $0 = H^n(X, \mathcal{O}_X) = H^n(X, F_*\mathcal{O}_X) = H^n(X, \mathcal{O}_X \oplus \mathcal{O}_X(a_2) \oplus \cdots \oplus \mathcal{O}_X(a_{p^n}))$.. By Serre duality this is the same as the vector space dual of $\oplus H^0(X, \mathcal{O}_X(-n-1) \otimes \mathcal{O}_X(-a_i))$. Since this is zero, none of the $-a_i$ can be larger than n (and so none of the a_i can be smaller than $-n$). In conclusion, the a_i (for $i > 1$) must all satisfy $0 > a_i \geq -n$.

We begin with $X = \mathbb{P}^2$. We know that $F_*\mathcal{O}_{\mathbb{P}^2} = \mathcal{O}_X \oplus \mathcal{O}_X(a_2) \oplus \cdots \oplus \mathcal{O}_X(a_{p^2})$ for various a_i (the rank can be computed on $\mathbb{A}^2$). We recall that on $\mathbb{P}^2$, $h^0(\mathcal{O}_X(n)) = \dim_k H^0(X, \mathcal{O}_X(n)) = \binom{n+2}{2}$. Thus $h^0((F_*\mathcal{O}_X)(1)) = h^0(\mathcal{O}_X(p)) = \binom{p+2}{2}$.

On the other hand $h^0\left(\mathcal{O}_X(1) \oplus \mathcal{O}_X(a_2+1) \oplus \cdots \oplus \mathcal{O}_X(a_{p^2}+1)\right) = \binom{3}{2}+$ the number of a_i equal to -1. So, we consider

$$\binom{p+2}{2} - \binom{3}{2} = (p+2)(p+1)/2 - 3 = \frac{1}{2}p^2 + \frac{3}{2}p - 2.$$

In characteristic $p = 5$, the number of summands total is 25. We know that $a_1 = 0$, so there is 1 summand of the form $\mathcal{O}_X$. We also compute $\binom{p+2}{2} - \binom{3}{2} = 18$. This leaves us with 25-1-18 $= 6$ summands left, by our above work, these must all be equal to -2. We can also show it directly, which is what has to be done in higher dimensions.

Now we twist by (2). In this case, we have $h^0((F_*\mathcal{O}_X)(2)) = h^0(\mathcal{O}_X(2p)) = \binom{2p+2}{2} = (2p+2)(2p+1)/2$. On the other hand $\mathcal{O}_X(2) \oplus \mathcal{O}_X(1)^{\oplus 18}$ has a $\binom{2+2}{2} + ((p+2)(p+1)/2 - 3)\binom{3}{2}$ dimensional vector space of global sections. In characteristic $p = 5$, $h^0((F_*\mathcal{O}_X)(2)) = 66$ while $\binom{2+2}{2} + ((p+2)(p+1)/2)\binom{3}{2} = 60$. Thus there must be exactly 6 terms of the form $\mathcal{O}_X(-2)$.

Trying this same computation in characteristic 7 gives us the following.

- 1 copy of $\mathcal{O}_X$.
- 33 copies of $\mathcal{O}_X(-1)$.
- 15 copies of $\mathcal{O}_X(-2)$.

In general, there is

- 1 copy of $\mathcal{O}_X$.
- $\frac{1}{2}p^2 + \frac{3}{2}p - 2$ copies of $\mathcal{O}_X(-1)$.
- $\frac{1}{2}p^2 - \frac{3}{2}p + 1$ copies of $\mathcal{O}_X(-2)$.

One can check that these numbers add up to p^2.

For $X = \mathbb{P}^3$, we know that $h^0(\mathcal{O}_X(n)) = \binom{n+3}{3}$. Similar computations yield:

- 1 copy of $\mathcal{O}_X$

- $\frac{1}{6}p^3 + p^2 + \frac{11}{6}p - 3 = \frac{1}{6}(p+3)(p+2)(p+1) - 4$ copies of $\mathcal{O}_X(-1)$.
- $\frac{2}{3}p^3 - \frac{11}{3}p + 3 = \frac{1}{6}(2p+3)(2p+2)(2p+1) - (1)(10) - (\frac{1}{6}(p+3)(p+2)(p+1) - 4)(4)$ copies of $\mathcal{O}_X(-2)$
- $\frac{1}{6}p^3 - p^2 + \frac{11}{6}p - 1 = \frac{1}{6}(3p+3)(3p+2)(3p+1) - (1)(20) - (\frac{1}{6}(p+3)(p+2)(p+1) - 4)(10) - (\frac{1}{6}(2p+3)(2p+2)(2p+1) - (1)(10) - (\frac{1}{6}(p+3)(p+2)(p+1))(4))$ copies of $\mathcal{O}_X(-3)$

One again checks that the sum of these equals p^3.

I do not know of anything more general than this. It could easily be implemented into a computer if one wanted to do the check for any fixed p and n (possibly even for a generic p and fixed n). There also might be a better approach to this problem in the literature, but I didn't find it (except for the previously mentioned work of Thomsen).

1. Rational singularities

For about 40 years, rational singularities have been the gold standard of *nice* singularities. In particular, given any class of singular varieties, the first question people tend to ask is, "Does it have rational singularities?" We'll see today that rational singularities are certainly not so far from F-pure singularities.

Definition 1.1 (Watanabe)**.** Given a normal graded d-dimensional ring R with $R_0 = k$ and irrelevant ideal $\mathfrak{m} = R$, we define the *a-invariant* of R, as follows:

$$a(R) := \max\{n | (H^d_{\mathfrak{m}}(R))_n \neq 0\} = -\min\{n | (\omega_R)_n \neq 0\}.$$

Recall the following fact: If S is a standard $\mathbb{N}$-graded ring (again, you don't need standard) with irrelevant ideal $\mathfrak{m} = S_+$, with $Y = \operatorname{Proj} S$, then

$$(H^i_{\mathfrak{m}}(S))_n = H^{i-1}(Y, \mathcal{O}_Y(n)),$$

for $i > 1$. This fact is quite easy to check using Čech cohomology.

If R is an R-pure ring, then $H^d_{\mathfrak{m}}(R))_n = 0$ for $n > 0$. To see this, simply note that we have injective maps $F^e : (H^d_{\mathfrak{m}}(R))_n \to (H^d_{\mathfrak{m}}(R))_{p^e n}$ for all e and the right side vanishes for $e \gg 0$ (this is completely clear by what we wrote above in a standard graded ring by Serre vanishing). Therefore, if R is is F-split, then $a(R) \leq 0$.

Watanabe also proved the following.

Theorem 1.2 (Watanabe)**.** *If R is a normal graded ring finitely generated over $k = R_0$, then R has rational singularities if and only if R satisfies the following two conditions:*

(i) *$U = \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$ has rational singularities.*
(ii) *R is Cohen-Macaulay and $a(R) < 0$.*

Thus, it is obvious that there is a very close relationship between F-purity and rational singularities. Notice that I haven't defined Cohen-Macaulay or rational singularities.

1.1. Cohen-Macaulay rings.

Briefly recall the following definition.

Definition 1.3. A local ring $(R, \mathfrak{m})$ of dimension d is called Cohen-Macaulay if there is a regular sequence of length d on R. In other words, if $x_1, \dots, x_n$ is a list of elements of $\mathfrak{m}$ such that x_{i+1} is a regular element (non-zero divisor) on $R/(x_1, \dots, x_i)$ for all i. A scheme is called Cohen-Macaulay if all of its stalks are Cohen-Macaulay local rings.

Compare the notion of a regular sequence with the (weaker) notion of a (full) system of parameters.

Definition 1.4. Elements $x_1, \dots, x_n \in R$ (a local ring of dimension n) form a *full system of parameters* if $\sqrt{x_1, \dots, x_n} = \mathfrak{m}$.

Remark 1.5. In fact, in a Cohen-Macaulay local ring, any system of parameters is a regular sequence (so if you find a system of parameters that is not a regular sequence, the ring is not Cohen-Macaulay). See [BH93].

Example 1.6. The following rings are Cohen-Macaulay.

- Any reduced one dimensional ring (choose any non-zero divisor).
- Any regular ring (any set of minimal generators of the maximal ideal will work).
- Any hypersurface singularity, or more generally, a complete intersection (this is a ring cut out by part of a regular sequence in a regular ring, and so in particular a Cohen-Macaulay ring, choose some additional parameters completing the sequence).

However, the following ring is not Cohen-Macaulay.

- $k[x, y, u, v]/((x, y) \cap (u, v)) = k[x, y, u, v]/((xu, xv, yu, yv))$. To see this, first notice that $x - u$ is not a zero divisor (it doesn't vanish on either component). Modding out by $x - u$ gives us the following ring $T := k[x, y, v]/(x^2, xv, xy, yv)$. We simply have to convince ourselves that every element of the maximal ideal of this ring is a zero divisor but this is easy since x kills every element of the maximal ideal of T.

1.2. **The Homological viewpoint on Cohen-Macaulay, Gorenstein and $\mathbb{Q}$-Gorenstein conditions.** First we remind ourselves what the derived category $D^b_{\mathrm{coh}}(X)$ is. The objects are complexes of $\mathcal{O}_X$-modules with coherent cohomology and only finitely many places with non-zero cohomology. For example, if $f : Y \to X$ is proper, then $Rf_*\mathcal{O}_Y$[1] is an object of $D^b_{\mathrm{coh}}(X)$. The morphisms of $D^b_{\mathrm{coh}}(X)$ are more complicated, they are equivalence classes of morphisms (up to chain homotopy equivalence) where we also invert all the $\mathcal{O}_X$-modules

Definition 1.7. Given a scheme X, an object $\omega_X^{\bullet} \in D^b_{\mathrm{coh}}(X)$ is called a *dualizing complex* if it has finite injective dimension (in other words, it is quasi-isomorphic to a *FINITE* complex of injectives) and if $R\mathscr{H}\mathrm{om}^{\bullet}_{\mathcal{O}_X}(\omega_X^{\bullet}, \omega_X^{\bullet}) \cong \mathcal{O}_X$.

That fancy $R\,\mathrm{Hom}^{\bullet}_{\mathcal{O}_X}$ is some derived functor of Hom (ie, replace the second term by a complex of injectives, and apply the first operation term by term).

Generally speaking, if you have a short exact sequence, such as $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$, we do get something like a short exact sequence when applying a derived functor like Rf_* (where $f : Y \to X$ is a proper map of schemes). The output is called an exact triangle and is denoted by $Rf_*\mathscr{A} \longrightarrow Rf_*\mathscr{B} \longrightarrow Rf_*\mathscr{C} \xrightarrow{+1}$. Taking cohomology of each complex $Rf_*\mathscr{A}$, $Rf_*\mathscr{B}$ and $Rf_*\mathscr{C}$ yields the usual long exact sequence.

Remark 1.8. Dualizing complexes are unique up to shifting (you can shift any complex) and up to tensoring with invertible sheaves. See [Har66] for details.

[1] $Rf_*\mathcal{O}_Y$ is defined as follows. Take an injective resolution $\mathscr{I}^{\bullet}$ of $\mathcal{O}_Y$ and set $Rf_*\mathcal{O}_Y = f_*\mathscr{I}^{\bullet}$. A Čech resolution is fine too.

Remark 1.9. Any quasi-projective scheme has a dualizing complex. Also, any F-finite affine scheme has a dualizing complex. If $X \subseteq \mathbb{P}^n_k$ is a projective variety, $\omega_X^{\bullet}$ can be defined to be $R\mathscr{H}\mathrm{om}_{\mathcal{O}_{\mathbb{P}^n_k}}(\mathcal{O}_X, \wedge^n \Omega^1_{X/k})$. For a quasi-projective variety, simply localize. Such dualizing complexes are nice because they are "normalized" at each maximal ideal of X (in particular the cohomology of $\omega_X^{\bullet}$ generically only lives in degree $-\dim X$). In this case $h^{-d}(\omega_X^{\bullet})$ is called a *canonical module* for X and is denoted by ω_X. Again, if X is normal, then any divisor K_X such that $\mathcal{O}_X(K_X) \cong \omega_X$ is called a canonical divisor.

Definition 1.10. Suppose that R is a local ring with a dualizing complex $\omega_R^{\bullet}$ and a canonical module ω_R (for example, $R = S_{\mathfrak{q}}$ is the localization of a ring S that is normal and of finite type over a field k, the canonical module was constructed as $\omega_R := (\wedge^{\dim S}\Omega_{S/k})^{**}_{\mathfrak{q}}$).

- We say that R is *Cohen-Macaulay* if $\omega_X^{\bullet}$ is quasi-isomorphic to ω_X.
- We say that R is *quasi-Gorenstein*[2] if $\omega_R \cong R$ (in a non-local setting, this means that ω_X is locally free or equivalently, that K_X is a Cartier divisor).
- We say that R is $\mathbb{Q}$-*Gorenstein* if there exists an integer $n > 0$ such that nK_R is a Cartier divisor[3] (it is probably best to assume that R is normal, unless you are already familiar with the theory of Weil-divisors on non-normal varieties).
- We say that R is *Gorenstein* if it is Cohen-Macaulay and quasi-Gorenstein.

If R is not-necessarily local, we say that R is

Cohen-Macaulay/quasi-Gorenstein/$\mathbb{Q}$*-Gorenstein/Gorenstein*

if $R_{\mathfrak{q}}$ satisfies the same property for every $\mathfrak{q} \in \operatorname{Spec} R$.

Remark 1.11. Notice that $\mathbb{Q}$-Gorenstein rings are not necessarily Cohen-Macaulay (although some authors make different definitions).

Proposition 1.12. *Every regular ring is Gorenstein, and furthermore, every complete intersection is also Gorenstein (in particular, a hypersurface singularity is Gorenstein). Most generally, if R is Gorenstein/Cohen-Macaulay, then so is R/f for any regular element $f \in R$ (the converse holds locally on R).*

Proof. See for example [BH93]. ☐

Example 1.13. The curve singularity $R = k[x, y, z]/(xy, xz, yz)$ is Cohen-Macaulay but not Gorenstein. To check that it is Cohen-Macaulay, simply notice that it is reduced and 1-dimensional. To see that it is not Gorenstein, we take a regular element $f = x + y - z$ and notice that $R/f = k[x, y]/(xy, x^2 + xy, xy + y^2) = k[x, y]/(x^2, xy, y^2)$. So we need merely check whether R/f is Gorenstein. By [BH93, Exercise 3.2.15], it is enough to find non-zero ideals I and J such that $I \cap J = 0$. But that is easy $I = (x)$, $J = (y)$.

Finally, we also state Grothendieck duality.

Theorem 1.14. [Har66] *Given a map of schemes $f : Y \to X$ of finite type, there exists a functor $f^! : D^b_{\mathrm{coh}}(X) \to D^b_{\mathrm{coh}}(Y)$. If furthermore, f is proper then one has the following:*

(i) $R\mathscr{H}\mathrm{om}^{\bullet}_{\mathcal{O}_X}(Rf_*\mathscr{F}^{\bullet}, \mathscr{G}^{\bullet}) \cong Rf_*R\mathscr{H}\mathrm{om}^{\bullet}_{\mathcal{O}_Y}(\mathscr{F}^{\bullet}, f^!\mathscr{G}^{\bullet})$ *where* $\mathscr{F}^{\bullet}, \mathscr{G}^{\bullet} \in D^b_{\mathrm{coh}}(X)$.

(ii) $f^!\omega_X^{\bullet}$ *is a dualizing complex for Y (denoted now by $\omega_Y^{\bullet}$).*

[2]This is also sometimes called begin 1-*Gorenstein*

[3]You can have Weil divisors such that a power is a Cartier divisor, consider a ruling on the quadric cone $xy - z^2$.

(iii) *If $f : Y \to X$ is a finite map (for example, a closed immersion), $f^!$ is identified with $R\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(f_*\mathcal{O}_Y, \_\_)$ (viewed then as a module on Y).*

We will also use Kodaira vanishing and a relative version, Grauert-Riemenschneider vanishing.

Theorem 1.15 (Kodaira Vanishing)**.** *Suppose that X is a smooth variety of characteristic zero and $\mathscr{L}$ is an ample line bundle on X. Then $H^i(X, \omega_X \otimes \mathscr{L}) = 0$ for $i > 0$ or dually, $H^i(X, \mathscr{L}^{-1}) = 0$ for $i < \dim X$.*

Theorem 1.16. [GR70] *Suppose that $\pi : \widetilde{X} \to X$ is a proper map of algebraic varieties in characteristic zero with $\widetilde{X}$ smooth. Then $R^i\pi_*\omega_{\widetilde{X}} = 0$ for $i > 0$.*

Remark 1.17. Both of these theorems *FAIL* in characteristic $p > 0$.

References

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[Har66] R. Hartshorne: *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. With an appendix by P. Deligne. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR0222093 (36 #5145)