F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 8-31-2010

KARL SCHWEDE

1. Flatness of Frobenius implies regular

Today, we'll complete the proof that having a flat Frobenius map implies that X is regular (a result of Kunz).

Theorem 1.1. *Suppose that X is a scheme, then R is regular if and only if $F^e_* \mathcal{O}_X$ is flat as an $\mathcal{O}_X$-module for some $e > 0$.*

Proof. We'll need several lemmas, but let us sketch the proof first. The statement is local so we may assume that $X = \operatorname{Spec} R$ where $(R, \mathfrak{m})$ is a local ring. Write $\mathfrak{m} = (x_1, \dots, x_n)$ where the x_i are a minimal system of generators. Our goal is to show that $n = \dim R$.

First observe that it is harmless to replace e by ne for any integer $n > 0$. Unlike what I said in class, the proof works fine for non-algebraically closed residue fields.

Step 1. $\mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2$ is a free R-module.
Step 2. Apply lemmas of Lech to conclude that $l_R(R/\mathfrak{m}^{[p^e]}) = p^{ne}$ for all $p \in N$.
Step 3. Assume R is complete and write $R = S/\mathfrak{a} = k[[x_1, \dots, x_n]]/\mathfrak{a}$. Then notice that $l_S(S/\mathfrak{m}_S^{[p^e]}) = p^{ne}$ for all $e \geq 0$. But this implies that $\mathfrak{a} = 0$ and so $R = S$. This actually completes the proof of step 3.

We begin with the proof of step 1.

$$F_* \mathfrak{m}^{[p^e]}/(\mathfrak{m}^{[p^e]})^2 = (\mathfrak{m}/\mathfrak{m}^2) \otimes_R F_* R = (\mathfrak{m}/\mathfrak{m}^2) \otimes_(R/\mathfrak{m}) F_*(R/\mathfrak{m}^{[p^e]})$$

because of flatness of $F_* R$ over R. But the right side is a free $F_*(R/\mathfrak{m}^{[p^e]})$-module. This implies that the (minimal set of) generators $x_1^{p^e}, \dots, x_n^{p^e}$ of $\mathfrak{m}^{[p^e]}$ are *Lech-independent.*

Definition 1.2. That a sequence of elements $f_1, \dots, f_n \in R$ is called *Lech-independent* if for any $a_1, \dots, a_n \in R$ such that $a_1 x_1^{p^e} + \dots + a_n x_n^{p^e} = 0$, then $a_i \in \mathfrak{m}^{[p^e]}$.

We now begin step 2. For this, we begin with a Lemma.

Lemma 1.3. [Lec64, Lemma 3] *If $f_1, \dots, f_n$ are Lech-independent elements and $f_1 \in gR$ for some $g \in R$, then $g, f_2, \dots, f_n$ is also Lech-independent. Furthermore, $(f_2, \dots, f_n) : g \subseteq (f_1, \dots, f_n)$*

Proof. Write $f_1 = gh$. Suppose $a_1 g + \dots + a_n f_n = 0$ multiplying the equation through by h implies that $a_1 \in (f_1, \dots, f_n) \subseteq (g, \dots, f_n)$ (this also proves the second statement of the theorem). Say $a_1 = b_1 f_1 + \dots + b_n f_n$. Plugging this in, we get that

$$0 = (b_1 f_1 + \dots + b_n f_n) g + a_2 f_2 + \dots + a_n f_n = b_1 g f_1 + (b_2 g + a_2) f_2 + \dots + (b_n g + a_n) f_n.$$

Therefore, $b_i g + a_i \in (f_1, \dots, f_n) \subseteq (g, f_2, \dots, f_n)$ for $i \geq 2$ and so $a_i \in (g, f_2, \dots, f_n)$ for $i \geq 2$ as desired. □

This lemma, combined with the fact that $x_1^{p^e}, \dots, x_n^{p^e}$ are Lech-independent, proves that $x_1^{\alpha_1}, \dots, x_n^{\alpha_n}$ are Lech-independent for $\alpha_i \leq p^e$ (or basically for any α_i since we can make e bigger). We now need another Lemma.

Lemma 1.4. [Lec64, Lemma 4] *If $f_1, \dots, f_n$ are Lech-independent and $f_1 = gh$. Then*

$$l_R\left(R/(f_1, \dots, f_n)\right) = l_R\left(R/(g, f_2, \dots, f_n)\right) + l_R\left(R/(h, f_2, \dots, f_n)\right).$$

Proof. First notice that

$$l_R\left(R/(f_1, \dots, f_n)\right) = l_R\left(R/(g, f_2, \dots, f_n)\right) + l_R\left((g, f_2, \dots, f_n)/(f_1, \dots, f_n)\right)$$

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However,

$$(g, f_2, \dots, f_n)/(f_1, \dots, f_n) = (gR + (f_1, \dots, f_n))/(f_1, \dots, f_n) \cong R/((f_1, \dots, f_n) : gR)$$

. We certainly know that $(f_1, \dots, f_n) : gR \supseteq (h, f_2, \dots, f_n)$ and we will show the converse inclusion. Suppose then that $ag = a_1 f_1 + \cdots + a_n f_n$, then $(a_1 h - a)g + a_2 f_2 + \cdots + a_n f_n = 0$, so that the $a_1 h - a \in (f_2, \dots, f_n) : g \subseteq (f_1, \dots, f_n)$. But then $a_1 h - a = b_1 f_1 + \cdots + b_n f_n = b_1 gh + \cdots + b_n f_n$ which implies that $a \in (h, b_2, \dots, b_n)$. ☐

We will explain how this lemma implies (inductively) that $l_R(R/\mathfrak{m}^{[p^e]}) = p^{ne}$ as desired. We will show that $l_R\left(R/(x_1^{\alpha_1}, \dots, x_n^{\alpha_n})\right) = \alpha_1 \cdot \alpha_2 \cdot \cdots \cdot \alpha_n$ by induction on $\sum_i \alpha_i$. The base case is obvious.

If $\alpha_i > 1$, by the previous lemma, we know that

$$\begin{aligned} & l_R\left(R/(x_1^{\alpha_1}, \dots, x_n^{\alpha_n})\right) \\ = {} & l_R\left(R/(x_1^{\alpha_1}, \dots, x_{i-1}^{\alpha_{i-1}}, x_i^1, x_{i+1}^{\alpha_{i+1}}, \dots, x_n^{\alpha_n}\right) + l_R\left(R/(x_1^{\alpha_1}, \dots, x_{i-1}^{\alpha_{i-1}}, x_i^{\alpha_i - 1}, x_{i+1}^{\alpha_{i+1}}, \dots, x_n^{\alpha_n})\right) \\ = {} & (\alpha_1 \cdot \cdots \cdot \alpha_{i-1} \cdot 1 \cdot \alpha_{i+1} \cdot \cdots \cdot \alpha_n) + (\alpha_1 \cdot \cdots \cdot \alpha_{i-1} \cdot (\alpha_i - 1) \cdot \alpha_{i+1} \cdot \cdots \cdot \alpha_n) \\ = {} & \alpha_1 \cdot \cdots \cdot \alpha_n \end{aligned}$$

which completes the induction.

Finally, we do step 3 (which we already did). ☐

2. Criteria for local Frobenius splitting I (Fedder's criteria)

Today, we'll learn about a result called for the second statement, assume that $ag + a_2 f_2 + \cdots + a_n f_n = 0$, so Fedder's criteria for local Frobenius splitting. We'll also explore Frobenius splitting of projective varieties vs Frobenius splitting of graded rings.

First local behavior. Suppose that S is an F-finite regular ring such that $F_* S$ is a free S-module (for example, this happens if S is local). Write $R = S/I$. Suppose that $\phi : F_*^e R \to R$ is R-linear. Consider the following diagram where the vertical arrows are the natural quotients:

$$\begin{array}{ccc} F_*^e S & \overset{\psi}{\dashrightarrow} & S \\ \downarrow & & \downarrow \\ F_*^e R & \underset{\phi}{\longrightarrow} & R \end{array}$$

Because $F_*^e S$ is free and thus projective, there exists a $F_*^e S$-module map ψ as labelled in the diagram (which makes the diagram commute). This map is not unique! If we further assume that S is local, then if ϕ is surjective, then so must be ψ (since if $\psi(S) \subseteq \mathfrak{m}_S$, then $\phi(S/I) \subseteq \mathfrak{m}_S/I = \mathfrak{m}_R \subsetneq R$.

Lemma 2.1. *With the notation as above, if R has a Frobenius splitting $\phi : F^e_* R \to R$ (ie, an R-linear map that sends 1 to 1), then there is a Frobenius splitting ψ' on S which also induces a (possibly different) Frobenius splitting on R as in the diagram above.*

Proof. We already saw the existence of a map $\psi : F^e_* S \to S$ which is surjective. Suppose that $\psi(x) = 1$. Then consider the map $\psi : F^e_* S \to S$ defined by the rule $\psi'(\_\_) = \psi(x \cdot \_\_)$, this is clearly a splitting. This map still induces a map on R (defined by $\phi'(\_\_) = \phi(\bar{x} \cdot \_\_)$) and it is a splitting since ψ' is). □

This suggests that in order to study the (possible) existence of F-splittings of R it might be good to study the splittings on S which induce splittings on R. First suppose that S is a regular local ring, let us study the maps $\phi \in \mathrm{Hom}_S(F^e_* S, S)$. To do this, I'd like to describe a little bit of duality for a finite map (Frobenius being the finite map).

In order to do this, we need a little bit of theory. So let's quickly review (Grothendieck) duality for a finite map.

Definition 2.2. Suppose that R is a local ring with a normalized dualizing complex $\omega_R^{\bullet}$. Then the *canonical module* ω_R of R is $\mathcal{H}^{-\dim R}(\omega_R^{\bullet})$. A canonical module on an arbitrary ring/scheme is a module whose localization is isomorphic the canonical module at every prime/point.

Somewhat more explicitly, we can define the canonical module of R as follows. If X is a normal irreducible scheme of (essentially) finite type over a field. One can define ω_X as follows:

$$\omega_X = \left(\wedge^{\dim X} \Omega^1_{X/k}\right)^{**}.$$

Here the symbol ** means apply the functor $\mathrm{Hom}_R(\_\_, R)$ twice.

Definition 2.3. A divisor K_X on a normal scheme X such that $\mathcal{O}_X(K_X) \cong \omega_X$ is called a *canonical divisor*.

Canonical divisors are divisor classes on varieties over fields. This is much more ambiguous on general schemes since ω_X can be twisted by any line bundle and still be a canonical module (we only defined it locally).

Theorem 2.4. [Har66] *Let $R \subseteq S$ be a finite inclusion of rings with dualizing complexes and that ω_R is a canonical module for R. Then:*

(i) $\mathrm{Hom}_R(S, \omega_R)$ *is a canonical module for S and if we are working with varieties of finite type over a field, we may assume that the canonical module constructed in this way for S, agrees with the one obtained by taking wedge-powers of $\Omega_{X/k}$.*
(ii) *If N is an S-module, then we have an isomorphism of S-modules* $\mathrm{Hom}_R(N, \omega_R) \cong \mathrm{Hom}_S(N, \mathrm{Hom}_R(S, \omega_R)) \cong \mathrm{Hom}_S(N, \omega_S)$.

Remark 2.5. The functor $\mathrm{Hom}_R(S, \_\_)$ is often called $f^{\flat}$ or $f^!$ where $f : \mathrm{Spec}\, S \to \mathrm{Spec}\, R$ is the induced map.

We will apply this theorem to the case of the Frobenius map.

Corollary 2.6. *Suppose that X is a normal scheme of essentially finite type over an F-finite field (or $X = \operatorname{Spec} R$ where R is an F-finite normal local ring). Then $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X((1-p^e)K_X)$.*

Proof. Let U denote the regular locus of X so that $X \setminus U$ is codimension 2 or higher. By basic facts about the reflexive sheaves, see for example [Har94], it is enough to show this isomorphism with X replaced by U (in other words, we may assume that X is regular). We may write

$$\begin{aligned} &\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \\ \cong\ &\mathscr{H}\mathrm{om}_{\mathcal{O}_X}((F^e_* \mathcal{O}_X) \otimes \mathcal{O}_X(K_X), \mathcal{O}_X(K_X)) \\ \cong\ &\mathscr{H}\mathrm{om}_{\mathcal{O}_X}((F^e_* \mathcal{O}_X(p^e K_X)), \mathcal{O}_X(K_X)) \\ \cong\ &\mathscr{H}\mathrm{om}_{F^e_* \mathcal{O}_X}(F^e_* \mathcal{O}_X(p^e K_X), F^e_* \mathcal{O}_X(K_X)) \\ \cong\ &F^e_* \mathcal{O}_X((1-p^e)K_X). \end{aligned}$$

The funny hypotheses at the start of this proof are there to insure that $s\operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X(K_X))$ is isomorphic to $\mathcal{O}_X(K_X)$ (and not some other canonical module). □

This greatly restricts which varieties can be globally Frobenius split.

Corollary 2.7. *Suppose that X is a Frobenius split variety, then $H^0(X, \mathcal{O}_X(-nK_X)) \neq 0$ for some $n > 0$. In particular, X cannot be projective and of general type.*

Proof. If X is Frobenius split then $\phi \in \operatorname{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X) \cong \mathcal{O}_X((1-p^e)K_X)$ is non-zero for some ϕ. In fact, one can take $e = 1$ and so $n = p - 1$. □

Another interesting conclusion of this is the following.

Corollary 2.8. *Suppose that $X = \operatorname{Spec} R$ where R is a normal F-finite local ring. If $\mathcal{O}_X((1-p^e)K_X)$ is locally free, then $\mathcal{O}_X((1-p^e)K_X)$ is also locally free and thus isomorphic to $\mathcal{O}_X$ (this happens for example if R is Gorenstein). In particular, $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ is a cyclic $F^e_* \mathcal{O}_X$-module. A $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ which generates $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ is called a* generating homomorphism.

Example 2.9. If $X = \operatorname{Spec} k[x_1, \dots, x_n]$, then the map which sends $(x_1 \dots x_n)^{p^e - 1}$ to 1 and the other relevant monomials to zero, is a "generating map". In the local case, there are other generating maps as well (send some of the other monomials to non-zero things).

Now we need some notation.

References

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