F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 12/9-2010

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1. Fujita's conjecture

We begin with a discussion of Castlenuovo-regularity, see [Laz04, Section 1.8].

Definition 1.1. Let $\mathscr{F}$ be a coherent sheaf on a projective variety X with a given ample line bundle $\mathscr{A} = \mathcal{O}_X(A)$ which is generated by global sections. A coherent sheaf $\mathscr{F}$ on X is called *m-regular* with respect to $\mathscr{A}$ if

$$H^i(X, \mathscr{F} \otimes \mathscr{A}^{\otimes(m-i)}) = 0$$

for $i > 0$.

Theorem 1.2 (Mumford)**.** [Laz04, Theorem 1.8.5] *With notation as above, suppose that $\mathscr{F}$ is an m-regular sheaf. Then $\mathscr{F} \otimes \mathscr{A}^m$ is globally generated.*

Example 1.3. [Laz04, Example 1.8.23]. Suppose that X is a smooth (or log canonical) n-dimensional variety of characteristic zero and $\mathscr{A} = \mathcal{O}_X(A)$ is an ample line bundle on X.

Now, for each $k \geq$ set $\mathscr{F}_k = \omega_X$. Clearly,

$$H^i(X, \mathscr{F}_k \otimes A^{\otimes(n+k-i)}) = 0$$

by Kodaira vanishing for any $k \geq 0$ and any $i > 0$. Thus ω_X is $n + k$-regular for all $k \geq 1$.

Applying the theorem above implies that $\mathcal{O}_X(K_X + (n + k)A)$ is globally generated for any $k \geq 1$.

Conjecture 1.4 (Fujita)**.** [Fuj87] *Suppose that $\mathscr{A}$ is an ample line bundle on a smooth n-dimensional variety X. Then:*

(i) *$\omega_X \otimes \mathscr{A}^{n+1+k}$ is globally generated for $k \geq 0$.*
(ii) *$\omega_X \otimes \mathscr{A}^{n+2+k}$ is very ample for $k \geq 0$.*

While we showed that (i) holds under the hypotheses that $\mathscr{A}$ is globally generated, condition (ii) also holds under the same condition, see [Laz04]. There a numerous refinements of this theorem by many authors including Angehrn, Demailly, Helmke, Kawamata, Kollár, Lazarsfeld, Seunghun Lee, Matsushita, Siu, Tsuji, and many others and has spawned much research in regards to Seshadri constants. It has been shown in characteristic zero in up through dimension 4, notably in [Rei88], [EL93], [Kaw97].

In characteristic $p > 0$, much less is known.

Theorem 1.5. [Smi97], *cf* [Har05] *Suppose that X is a variety over $k = \bar{k}$. If X is only F-rational and $\mathscr{A}$ is globally generated then (i) holds in characteristic $p > 0$.*

The proof uses tight-closure methods, and we will prove it shortly.

Theorem 1.6. [Kee08] *Suppose that X is a variety over $k = \bar{k}$. If X is smooth and $\mathscr{A}$ is globally generated, then (ii) holds in characteristic $p > 0$.*

The proof uses Arapura's theory of Frobenius amplitude, which can be thought of as a means to measure positivity of line bundles and other sheaves in positive characteristic.

We now turn to the proof of (i) in positive characteristic, we follow Hara's approach from [Har05]. First recall the following definition.

Definition 1.7. Given an ideal $\mathfrak{a}$ in a ring R and an integer $t > 0$, the *test submodule* $\tau(\omega_R, \mathfrak{a}^t)$ is defined to be the unique smallest submodule $J \subseteq \omega_R$ such that

$$\Phi^e_R(F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil} J) \subseteq J$$

where $\Phi_R : F_*\omega_R \to R$ is the dual of Frobenius. It is also harmless to replace $\mathfrak{a}^{\lceil t(p^e-1)\rceil}$ by $\overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil}}$ in the previous equation.

Given an appropriate test element $c \in R$, we still have

$$\tau(\omega, \mathfrak{a}^t) = \sum_{e \geq 0} \Phi^e_R(F^e_* c\overline{\mathfrak{a}^{\lceil tp^e\rceil}}\omega_R)$$

Any difference between using integral closures or not (or tp^e vs $t(p^e-1)$ can be absorbed into the c-term.

Lemma 1.8. [Har05, Proposition 2.4], [HT04] *Assume that R is a $\mathbb{N}$-graded ring of dimension $d \geq 1$ and further suppose that R has a graded system of parameters in degree 1. Set $\mathfrak{m} = R_+$. Then if $l \geq 0$ is an integer, we have*

$$\tau(\omega_R, \mathfrak{m}^{l+d-1}) = \overline{\mathfrak{m}^l}\tau(\omega_R, \mathfrak{m}^{d-1}).$$

Proof. The proof is essentially the same as a proof in [BSTZ10]. Choose $\mathfrak{a}$ to be the ideal generated by our given system of parameters noting that $\overline{\mathfrak{a}} = \mathfrak{m}$ (in particular, it is generated by d-elements). We consider the dual of Frobenius, $\Phi_R : F_*\omega_R \to \omega_R$. We then note the following equality,

$$\mathfrak{m}^{p^n(l+d-1)} = \mathfrak{a}^{p^n(l+d-1)} = \mathfrak{a}^{p^n l}\mathfrak{a}^{p^n(d-1)} = (\mathfrak{a}^{[p^n]})^l \mathfrak{a}^{p^n(d-1)}.$$

Then for some $w > 0$ and appropriate $0 \neq c \in R$ we have:

$$\tau(\omega_R, \mathfrak{a}^{l+d-1}) = \sum_{n=1}^{w} \Phi^n_R(F^n_* \mathfrak{a}^{(l+d-1)p^n} c\omega_R), \text{ and}$$

$$\tau(\omega_R, \mathfrak{a}^{l+d-1}) = \sum_{n=1}^{w} \Phi^n_R(F^n_* \overline{\mathfrak{a}^{(l+d-1)p^n}} c\omega_R), \text{ and}$$

$$\tau(\omega_R; \mathfrak{a}^{d-1}) = \sum_{n=1}^{w} \Phi^n_R(F^n_* \mathfrak{a}^{(d-1)p^{ne}} c\omega_R).$$

However,

$$\begin{aligned}
\tau(\omega_R, \mathfrak{a}^{l+d-1}) &= \sum_{n=1}^{w} \Phi_R^n(F_*^n \mathfrak{a}^{(l+d-1)p^n} c\omega_R) \\
&= \sum_{n=1}^{w} \Phi_R^n(F_*^n (\mathfrak{a}^{[p^n]})^l \mathfrak{a}^{p^n(d-1)} c\omega_R) \\
&= \sum_{n=1}^{w} \Phi_R^n(F_*^n \overline{(\mathfrak{a}^l)}^{[p^n]} \mathfrak{a}^{p^n(d-1)} c\omega_R) \\
&= \overline{(\mathfrak{a}^l)} \sum_{n=1}^{w} \phi_n(F_*^n \mathfrak{a}^{p^n(d-1)} c\omega_R) \\
&= \overline{\mathfrak{m}^l}\tau(\omega_R; \mathfrak{a}^{d-1})
\end{aligned}$$

as desired. □

Lemma 1.9. [Har05, Lemma 2.6] *Suppose that R is a d-dimensional normal graded ring over a perfect field $k = R_0$ of characteristic $p > 0$ with $\mathfrak{m} = R_+$ and also that R has a system of parameters of degree 1. Suppose further that R is F-rational on the punctured spectrum. Then*

$$\tau(\omega, \mathfrak{m}^l) = [\omega_R]_{>l}$$

for $l \gg 0$.

Proof. We will work in the Matlis dual world. The Matlis dual of $\omega_R/\tau(\omega, \mathfrak{m}^l)$ is $0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)}$ and so we want to show that

$$0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)} = H^d_\mathfrak{m}(R)_{\geq -l}$$

Recall that $0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)}$ is the set of elements $z \in H^d_\mathfrak{m}(R)$ such that there exists $0 \neq c \in R$ satisfying $c\overline{\mathfrak{m}^{lp^e}} z^{p^e} = 0 \in H^d_\mathfrak{m}(R)$.

So we have two containments to show. First suppose that $z \in H^d_\mathfrak{m}(R)_{\geq -l}$. Thus $\mathfrak{m}^{lp^e} z^{p^e} \in H^d_\mathfrak{m}(R)_{\geq 0}$, but $H^d_\mathfrak{m}(R)_{\geq 0}$ has finite length and so there is a non-zero element of R which annihilates it, which implies $z \in 0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)}$.

The reverse containment is somewhat more involved. First note that because $\omega_R/\tau(\omega, \mathfrak{m}^l)$ has support at the maximal ideal, $0^{*\mathfrak{m}^l}_{H^d_\mathfrak{m}(R)}$ has finite length. This implies that the Frobenius map

$$F^e : [H^d_\mathfrak{m}(R)]_{<-l} \to [H^d_\mathfrak{m}(R)]_{<-p^e l}$$

is injective for $l \gg 0$.

Choose $0 \neq z \in [H^d_\mathfrak{m}(R)]_{<-l}$. Therefore, $\lim_{e\to\infty} \deg(z^{p^e}) + lp^e = -\infty$.

Claim 1. *For $e \gg 0$, there exists a sequence of $c_e \in R$ such that $\lim_{e\to\infty} \deg(c_e) = \infty$ and such that $c_e R_{p^e l} z^{p^e} \neq 0$.*

Proof. The socle of $H^d_\mathfrak{m}(R)$ is the set of elements of $H^d_\mathfrak{m}(R)$ annihilated by $\mathfrak{m}$. This is a module of finite length since its Matlis dual is $\omega_R/(\mathfrak{m}\omega_R)$. To see this, given a set of generators y_i of $\mathfrak{m}$, the socle is the kernel of $H^d_\mathfrak{m}(R) \to \oplus y_i H^d_\mathfrak{m}(R)$. Matlis duality gives the claim. Likewise the module of elements of $H^d_\mathfrak{m}(R)$ annihilated by $R_{\geq n}$ is also finite length for any n.

Now, $R_{p^e l} z^{p^e}$ is non-zero for $e \gg 0$ because if it was zero, then $R_{p^e l-1} z^{p^e}$ would be in the socle or zero. Inductively, this is ridiculous. Thus we can find c_e satisfying the desired properties. $\square$

Using the fact that the degrees of c_e are increasing, it then follows (by arguments I won't repeat here, see the citation for more descriptions, or [Sch08]) that c_e is a "test element" for $e \gg 0$. We also know that $e \gg 0$, $c_e \overline{\mathfrak{m}}^{p^e l} z^{p^e} = c_e R_{p^e l} z^{p^e} \neq 0$, which implies that $z \notin 0^{*\mathfrak{m}^l}_{H^d_{\mathfrak{m}}(R)}$. This completes the proof. $\square$

We need one more lemma.

Lemma 1.10 (Smith)**.** *With notation as above, $\omega_X \otimes \mathscr{L}^{\otimes m}$ is globally generated if $[\omega_R]_l = R_{l-m}[\omega_R]_m$ for all $l \gg 0$.*

Proof. Suppose first the condition is satisfied, but that $\omega_X \otimes \mathscr{L}^{\otimes m}$ is not globally generated. In particular, the global sections of $\omega_X \otimes \mathscr{L}^{\otimes m}$ all vanish on some closed subvariety. But then $R_{l-m}[\omega_R]_m$ vanishes on that same subvariety for $l \gg 0$. $\square$

Now we turn to our main result of this section: Suppose that $\mathscr{A}$ is an ample and globally generated line bundle on a smooth n-dimensional variety X. Then $\omega_X \otimes \mathscr{A}^{n+1+k}$ is globally generated for $k \geq 0$.

Proof of Theorem 1.5. This is taken from [Har05, Theorem 2.1]. Set $R = R(X, \mathscr{A})$ and set $d = n + 1 = \dim X + 1 = \dim R$. As before, set $\mathfrak{m} = R_+$ and observe that $\overline{\mathfrak{m}^l} = R_{\geq l}$.

Now, we have the following inclusions for $l \gg 0$:

$$R_{l-d}[\omega_R]_{d-1} \subseteq [\omega_R]_{>l-1} = \tau(\omega, \mathfrak{m}^{l-1}) = R_{\geq l-d}\tau(\omega, \mathfrak{m}^{d-1}) \subseteq R_{\geq l-d}[\omega_R]_{>d-1}$$

This completes the proof. $\square$

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