F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 12/7-2010

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1. Kodaira-type vanishing in characteristic $p > 0$

First we recall Kodaira's vanishing theorem.

Theorem 1.1. [Kod53] *Suppose that X is a smooth projective variety of dimension n, characteristic zero, and H is an ample divisor on V, then*

$$H^i(X, \mathcal{O}_X(-H)) = 0$$

for $i = 0, 1, \ldots, n-1$. Dually, $H^i(X, \omega_X(H)) = 0$ for $i > 0$ (this dual version is equivalent as long as the variety is Cohen-Macaulay, which holds for example for normal surfaces).

This was known previously for surfaces, [Zar95]. It fails in characteristic zero for arbitrarily singular varieties (although it holds for normal surfaces), see for example [AJ89].

This result is also false in characteristic $p > 0$. We begin with Mumford's example (which is singular).

Example 1.2. [Mum67, Example 6] Suppose that X_0 is a normal surface in characteristic $p > 0$ with an element $\alpha \in H^1(X_0, \mathcal{O}_{X_0})$ such that $F(\alpha) = 0$ (for example, $X = E \times \mathbb{P}^1$ where E is a supersingular elliptic curve).

Suppose that H_0 is an irreducible hyperplane section of X_0 and let $L_0 = \mathcal{O}_{X_0}(H_0)$. Choose a open covering U_i of X_0 that principalizes H_0 and represent α as $\{\alpha_{ij}\}$ in Čech cohomology and choose $g_i \in \Gamma(U_i, \mathcal{O}_{X_0})$ so that $\alpha_{ij}^p = g_i - g_j$. Suppose that $H_0|_{U_i} = V(h_i)$ for some $h_i \in \Gamma(U_i, \mathcal{O}_{X_0})$. Define an extension L of $K(X)$ by adjoining all roots of the equations:

$$z_i^p - h_i^p z_i = g_i$$

Note that then $g_i - z_i^p = -h_i^p z_i$. Define $\pi : X \to X_0$ to be the normalization of X_0 inside L, and set $H = \pi^* H_0$ (note, H is ample since π is finite).

Claim 1. *$\pi^*\alpha$ is contained in the subspace $H^1(X, \mathcal{O}_X(-H)) \subseteq H^1(X, \mathcal{O}_X)$ (note that $H^0(X, \mathcal{O}_X)$ surjects onto $H^0(H, \mathcal{O}_H)$).*

Proof. We set $V_i := \pi^{-1}(U_i)$. Now, $z_i \in \Gamma(V_i, \mathcal{O}_X)$ since z_i satisfies a monic equation with coefficients in $H^0(X_0, \mathcal{O}_{X_0})$. This implies that

$$\begin{aligned} \pi^*\alpha &= [\alpha_{ij}] \\ &= [\alpha_{ij} - z_i + z_j] \end{aligned}$$

so that

$$\begin{aligned}\left(\frac{\alpha_{ij} - z_i + z_j}{h_i}\right)^p &= \frac{\alpha_{ij}^p - z_i^p + z_j^p}{h_i^p} \\ &= \frac{(g_i - g_j) - z_i^p + z_j^p}{h_i^p} \\ &= \frac{(g_i - z_i^p) - (g_j - z_j^p)}{h_i^p} \\ &= -z_i + (h_j/h_i)^p z_j \\ &\in \Gamma(V_i \cap V_j, \mathcal{O}_X)\end{aligned}$$

But this implies that $\left[\frac{\alpha_{ij}-z_i+z_j}{h_i}\right] \in \Gamma(V_i \cap V_j, \mathcal{O}_X)$ which itself implies that $\alpha = [\alpha_{ij} - z_i - z_j] \in \Gamma(V_i \cap V_j, \mathcal{O}_X(H))$ and the claim follows. □

The result then follows by the following lemma.

Lemma 1.3. [Mum67, Lemma 5] *Let $\pi : X' \to X$ be a finite surjective morphism of normal varieties over $k = \overline{k}$ such that $K(X) \subseteq K(X')$ is separable. Suppose that $\alpha \in H^1(X, \mathcal{O}_X)$ is such that $F(\alpha) = 0$ and $0 = \pi^*\alpha \in H^(X', \mathcal{O}_{X'})$. Then $\alpha = 0$.*

Proof. As before, represent α as $\{\alpha_{ij}\}$ in Čech cohomology for some cover U_i of X. Again we have $\alpha_{ij}^p = g_i - g_j$ with $g_i \in \Gamma(U_i, \mathcal{O}_{X_0})$. Because $\pi^*(\alpha) = 0$ there exists functions $h_i \in \Gamma(\pi^{-1}(U_i), \mathcal{O}_{X'})$ such that $\pi^*(\alpha_{ij}) = h_i - h_j$. Therefore,

$$h_i^p - \pi^*(g_i) = h_j^p - \pi^*(g_j).$$

Thus there exists a $\beta \in \Gamma(X', \mathcal{O}_{X'})$ such that $f^*(g_i) = h_i^p + \beta$ for all i. This implies that $\pi^*(g_i) \in K(X')^p$, which implies that $g_i \in K(X)^p$ for all i since $K(X) \subseteq K(X')$ is separable. Write $g_i = f_i^p$, $f_i \in K(X)$, and then since X is normal, we have that $f_i \in \Gamma(U_i, \mathcal{O}_X)$. Then, $a_{ij} = f_i - f_j$ since $a_{ij}^p = g_i - g_j$. This implies $\alpha = 0$ as desired. □

Remark 1.4. While there is no guarantee that X is smooth,

We now discuss Kawamata-Viehweg vanishing in positive characteristic.

Theorem 1.5. [Kaw82], [Vie82] *Suppose that X is a normal projective algebraic variety over an algebraically closed field of characteristic zero, B an effective $\mathbb{Q}$-divisor on X and D a Cartier (or $\mathbb{Q}$-Cartier integral) divisor. Assume that (X, B) is Kawamata log terminal and that $H = D - (K_X + B)$ is ample. Then $H^i(X, D) = 0$ holds for an $i > 0$.*

We will show that many varieties fail this, at least if they are constructed out of bizarre curves, we follow [Xie07].

Definition 1.6. [Tan72] Suppose that C is a smooth curve and $f \in K(C)$. Define

$$n(f) = \deg\lfloor \frac{1}{p} D(df) \rfloor.$$

Here $D(df)$ is the divisor associated to $df \in \omega_C$. The *Tango invariant* of C is defined to be

$$n(C) = \max\{n(f) | f \in K(C), f \notin (K(C))^p\}.$$

A curve C is called a *Tango curve* if $n(C) > 0$.

Before continuing, I'd like to discuss why Hiroshi Tango considered this notion, we will not include the proof at this time.

Theorem 1.7. [Tan72] *Let C be a curve of genus $g > 0$ with Tango invariant $n(C)$, then:*

(i) *For any line bundle $\mathscr{L}$ such that $\deg L > n(C)$, the Frobenius map $H^1(C, \mathscr{L}^{-1}) \to H^1(C, F^*\mathscr{L}^{-1})$ is injective (dually, $H^0(C, (F_*\omega_C) \otimes \mathscr{L}^p) \to H^0(C, \omega_C \otimes \mathscr{L})$ is surjective).*

(ii) *If $n(X) > 0$, then there exists a line bundle $\mathscr{M}$ of degree $n(C)$ such that the Frobenius map $H^1(X, \mathscr{M}^{-1}) \to H^1(X, F^*\mathscr{M}^{-1})$ is not injective.*

Remark 1.8. The Tango invariant of $\mathbb{P}^1$ is -1.

Example 1.9. [Tan72] The following curve $x^3y + y^3z + z^3x = 0$ in $\mathbb{P}^2$ is a genus 3 smooth Tango curve in characteristic 3. The partial derivatives are z^3, x^3, y^3 and so it is indeed smooth. Choose $f = (x - y)/z \in K(C)$. At the point $(0, 0, 1)$, we see that f vanishes to order 1, and so f is not in $K(C)^3$. One can show that

$$D(df) = -3(0,0,1) - 3(1,0,0) + \sum_{\alpha\alpha^3=\alpha+1} \lambda(1-\alpha, -1, 1) + \text{ other positive terms.}$$

where $\lambda \geq 3$. $n(f) \geq 1$.

Assuming $f \notin (K(C))^p$, $df \neq 0$ so that $D(df) \sim K_C$ and has degree $2g-2$ where $g = g(C)$ is the genus of C. Also notice that $n(C) \leq \lfloor (2g-2)/p \rfloor$, thus $n(C) > 0$ implies that $g > 1$. There are many examples of Tango curves.

We have the following two short exactly sequences (just like we explored in the proof of Hara's lemma):

$$0 \to \mathcal{O}_C \to F_*\mathcal{O}_C \to \mathcal{B}^1 \to 0$$
$$0 \to \mathcal{B}^1 \to F_*\Omega_C \to \Omega_C \to 0$$

Here $\mathcal{B}^1$ is the image of $d : F_*\mathcal{O}_C \to F_*\Omega_C$.

Lemma 1.10. [Xie07] *With notation as above let L be a divisor on C, then $H^0(C, \mathcal{B}^1(-L)) = \{df | f \in K(C), D(df) \geq pL\}$. Furthermore, $n(C) > 0$ if and only if there exists an ample divisor L on C such that $H^0(C, \mathcal{B}^1(-L)) \neq 0$.*

Proof. Twisting the second equation above by $-L$ we get

$$0 \to \mathcal{B}^1(-L) \to F_*(\Omega_C(-pL)) \to \Omega_C(-L) \to 0.$$

Now, $H^0(C, \Omega_C(-pL)) = \{\omega \in \Omega_C | D(\omega) \geq pL\}$, so that

$$H^0(C, \mathcal{B}^1(-L)) = \{df | f \in K(C), D(df) \geq pL\}.$$

For the second statement, assume that $n(C) > 0$, thus there exists an $f_0 \in K(C)$ such that $n(f_0) = \deg\lfloor D(df_0)/p \rfloor > 0$. Let $L = \lfloor D(df_0)/p \rfloor$. Certainly $\deg L > 0$ and $D(df_0) \geq pL$ and so $df_0 \in H^0(C, \mathcal{B}^1(-L)) \neq 0$ as desired. The converse direction merely reverses this. ☐

Using Tango curves, Raynaud constructed a smooth counterexample to Kodaira vanishing in each characteristic. These ideas have recently been further explored by Xie, and we have the following theorem.

Theorem 1.11. [Xie07] *Suppose that C is a tango curve, then there exists a $\mathbb{P}^1$-bundle $f : X \to C$ an effective $\mathbb{Q}$-divisor B and an integral divisor D on X such that (X, B) is KLT (in fact, B has SNC support with coefficients < 1) and $H = D - (K_X + B)$ is ample but $H^1(X, D) = 0$.*

Proof. This is taken from [Xie07]. We choose a divisor L on C such that $\deg L > 0$ and $H^0(C, \mathcal{B}^1(-L)) \neq 0$. Set $\mathscr{L} = \mathcal{O}_C(L)$, we then obtain

$$0 \to H^0(C, \mathcal{B}^1(-L)) \to H^1(C, \mathscr{L}^{-1}) \to H^1(C, \mathscr{L}^{-p}).$$

Choose $\alpha \in H^0(C, \mathcal{B}^1(-L))$ with image $\overline{\alpha} \in H^1(C, \mathscr{L}^{-1}) \cong \operatorname{Ext}^1_C(\mathscr{L}, \mathcal{O}_C)$. Thus we obtain an extension

$$0 \to \mathcal{O}_C \to \mathscr{E} \to \mathscr{L} \to 0.$$

Apply F^* and obtain

$$0 \to \mathcal{O}_C \to F^*\mathscr{E} \to \mathscr{L}^p \to 0$$

which corresponds to the extension class of $F^*\overline{\alpha}$, but this class is zero...

Let $f : X = \mathbb{P}(\mathscr{E}) \to C$ be the $\mathbb{P}^1$ bundle over C, with associated $\mathcal{O}_X(1)$ and fiber G. The surjection $\mathscr{E} \to \mathscr{L} \to 0$ induces a section $\sigma : C \to X$ by [Har77, IV, Prop 2.6] with image E. Furthermore, $f^*\mathcal{O}_C = \mathcal{O}_X \cong \mathcal{O}_X(1) \otimes \mathcal{O}_X(-E)$ so that $\mathcal{O}_X(E) = \mathcal{O}_X(1)$. We use the fact the sequence above is split and then and obtain:

$$0 \to \mathcal{O}_C \to (F^*\mathscr{E}) \otimes \mathscr{L}^{-p} \to \mathscr{L}^{-p} \to 0.$$

Thus we have the composition

$$H^0(C, \mathcal{O}_C) \to H^0(C, (F^*\mathscr{E}) \otimes \mathscr{L}^{-p}) \to H^0(C, S^p(\mathscr{E}) \otimes \mathscr{L}^{-p}) \cong H^0(X, \mathcal{O}_X(p) \otimes f^*\mathscr{L}^{-p}).$$

Thus we have a section $t \in H^0(X, \mathcal{O}_X(p) \otimes f^*\mathscr{L}^{-p})$ (corresponding to the image of 1). Therefore, we have a curve C' on X with $\mathcal{O}_X(C') \cong \mathcal{O}_X(p) \otimes f^*\mathscr{L}^{-p}$.

Claim 2. *We claim that C' is smooth and also that $C' \cap E = \emptyset$.*

Proof. We won't work out the details, but only sketch some evidence. Certainly $C'.E = (pE - p(\deg L)G).E = pE^2 - p(\deg L)$ where E^2 is the degree of $\mathscr{E}$ which is clearly $\deg L$. Thus as long as C' is irreducible, the second claim is obvious.

In fact, E and C' both correspond to splittings onto distinct terms of the split exact sequence

$$0 \to \mathcal{O}_C \to F^*\mathscr{E} \to \mathscr{L}^p \to 0.$$

compare with [Har77, Chapter V, Exercise 2.2]. □

Choose c a rational number satisfying $1/p < c < 1$ such that $cp \notin \mathbb{Z}$. Set $q = \lfloor cp \rfloor - 1$, and note that $q \geq 0$. Set $B = cC'$ and $D = qE + f^*(K_C - qL)$. Then

$$\begin{aligned} H &= D - (K_X + B) \\ &\equiv (\lfloor cp \rfloor - 1)E + f^*(K_C - qL) - K_X - cC' \\ &\equiv (\lfloor cp \rfloor - 1)E + f^*(K_C - (\lfloor cp \rfloor - 1)L) - (-2E + f^*K_C - f^*L) - c(pE - pf * L) \\ &\equiv (\lfloor cp \rfloor + 1 - cp)E + (cp - \lfloor cp \rfloor)f^*L. \end{aligned}$$

In particular, E is relatively ample and thus H is also ample. Clearly (X, B) is KLT.

Now, we need to show that $H^1(X,D) \neq 0$. Now, $D.G \geq 0$, thus by [Har77, Lemma 2.4], $R^1 f_* \mathcal{O}_X(D) = 0$ and $f_*\mathcal{O}_X(D)$ is locally free. Then

$$\begin{aligned} & H^1(X,D) \\ &= H^1(C, f_*\mathcal{O}_X(D)) \\ &= H^0(C, (f_*\mathcal{O}_X(D))^\vee \otimes \omega_C)^\vee \\ &= H^0(C, (f_*\mathcal{O}_X(D - f^*K_C))^\vee)^\vee \\ &= H^0(C, \mathcal{O}_C(qE - qL)^\vee)^\vee \\ &= H^0(C, (S^q(\mathscr{E})^\vee \otimes \mathscr{L}^q))^\vee. \end{aligned}$$

Now $\mathscr{L}^q$ is a quotient of $S^q(\mathscr{E})$, so $\mathscr{L}^{-q}$ is a subsheaf of $S^q(\mathscr{E})^\vee$. Thus,

$$H^1(X,D)^\vee = H^0(C, S^q(\mathscr{E})^\vee \otimes \mathscr{L}^q) \supseteq H^0(C, \mathscr{L}^{-q} \otimes \mathscr{L}^q) = H^0(C, \mathcal{O}_C) = k$$

proving the theorem. □

Q. Xie also proves the following result:

Theorem 1.12. [Xie07] *If there is a counter-example to the Kawamata-Viehweg vanishing theorem on a ruled surface $f : X \to C$, then either C is a Tango curve or all sections are ample.*

He also conjectures the following:

Conjecture 1.13. *If there is a counter-example to the Kawamata-Viehweg vanishing theorem on a normal projective surface X, then there exists a dominant rational map f from X to a smooth projective Tango curve C.*

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