F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 12/2-2010

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1. Criteria for F-splitting of varieties

Last time we proved the following.

Proposition 1.1. [BK05, Proposition 1.3.7] *Let X be a nonsingular variety. Then the following map η is an isomorphism. The map η*

$$\eta : \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \to \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

is defined as follows: Working locally, fix a local generator ω for $\omega_{X,x}$. Furthermore, for $\psi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_{X,x}}(\omega_{X,x}, F_*\omega_{X,x})$ and $f \in \mathcal{O}_{X,x}$, we define $\eta(\psi)f$ to be the ω coefficient of $T(f\psi(\omega))$.

This is well defined and furthermore, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X, F_*\omega_X) & \xrightarrow{\eta} & \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \\ {\scriptstyle T}\downarrow & & \downarrow{\scriptstyle \text{eval at 1}} \\ \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X, \omega_X) & \xrightarrow{\kappa} & \mathscr{H}\mathrm{om}(\mathcal{O}_X, \mathcal{O}_X) \end{array}$$

where κ is the natural isomorphism.

Now, $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \cong F_* \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^*\omega_X, \omega_X) \cong F_*\omega_X^{1-p}$. This yields a *canonical* isomorphism:

$$\alpha : \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong F_*\omega_X^{(1-p)}$$

Theorem 1.2. [BK05, Theorem 1.3.8] [MR85] *The evaluation-at-1 map $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \to \mathcal{O}_X$ is identified the map*

$$\sigma : F_*\omega_X^{(1-p)} \to \mathcal{O}_X$$

defined locally by

$$\sigma(f(dt_1 \wedge \dots dt_n)^{1-p}) = S(f).$$

Therefore, $\phi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ splits the Frobenius map if and only if $\sigma(\alpha(\phi)) = 1$.

Proof. The diagram in the previous proposition proves exactly the first claim, and the second follows immediately. □

Corollary 1.3. *For a smooth X, given $\phi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$, if ϕ is a splitting, then the $\mathbf{t}^{p-1}\omega$ coefficient of $\alpha(\phi)$ is equal to 1 for every point $x \in X$. For a general normal complete X, $\phi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is a splitting if and only if the $\mathbf{t}^{p-1}\omega$ coefficient of $\alpha(\phi)$ is equal to 1 for some point $x \in X$*

Proof. The $\mathbf{t}^{p-1}(dt_1 \wedge \dots dt_n)^{1-p}$-coefficient of $\alpha(\phi)$ is the constant term of $\phi(1)$ in $\mathcal{O}_{X,x} \subseteq k[[t_1, \dots, t_n]]$. Thus if $\phi(1) = 1$, this is just 1. For the complete case, we know that $\phi(1)$ is an element of $k = H^0(X, \mathcal{O}_X)$, and so the $\mathbf{t}^{p-1}(dt_1 \wedge \dots dt_n)^{1-p}$-coefficient of $\alpha(\phi)$ is the only term that matters. $\square$

We now come to the main result of this section. An effective tool for determining if a given $\phi \in \mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong F_*\omega_X^{(1-p)}$ is a splitting.

Theorem 1.4. [BK05], [MR85] *Suppose that X is a normal complete variety of dimension n. If there exists a $s \in H^0(X, \omega_X^{(-1)})$ with associated divisor*

$$D = Y_1 + \dots Y_n + Z$$

where $Y_1, \dots, Y_n$ are prime divisors which intersect with SNC at a smooth closed point $x \in X$ and Z is an effective divisor not containing x, then X is Frobenius split by a splitting corresponding to $s^{p-1} \in H^0(X, \omega_X^{(1-p)})$ up to a unit.

More generally, if $s \in H^0(X, \omega_X^{(1-p)})$ is such that the divisor of s is $(p-1)(Y_1 + \dots + Y_n) + Z$ where the Y_1 are SNC at a closed point $x \in X$ and Z does not contain X, then the same result holds.

Proof. At $x \in X$, suppose that each Y_i is given by the vanishing of some $t_i \in \mathcal{O}_{X,x}$. Then the power series expansion of s^{p-1} is simply $t_1^{p-1} \dots t_n^{p-1} g (dt_1 \wedge \dots \wedge dt_n)^{1-p}$ where g is a formal power series not vanishing at the origin. In particular, the section $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ corresponding to s^{p-1} sends 1 to a non-zero constant in k. Multiplying by the inverse of that constant gives us our desired result. $\square$

Remark 1.5. It should be noted that the ϕ constructed above is compatible with all the Y_i's and Z, since the Δ_ϕ is exactly $Y_1 + \dots + Y_n + Z = \frac{(p-1)}{(p-1)}(Y_1 + \dots + Y_n + Z)$.

Corollary 1.6. *Suppose that X is a complete n-dimensional variety in characteristic zero and Δ is a $\mathbb{Q}$-divisor in characteristic zero such that $\Delta = Y_1 + \dots Y_n + Z$ where the Y_i are prime divisors which intersect with SNC at a smooth closed point $x \in X$ and Z is an effective divisor not containing x. Further suppose that $K_X + \Delta \sim_{\mathbb{Q}} 0$, then (X, Δ) is log canonical.*

Proof. Reduce to characteristic $p \gg 0$, then (X_p, Δ_p) is F-split and thus locally F-pure. This implies that (X, Δ) is log canonical. $\square$

2. Diagonal splitting

Definition 2.1. [RR85] Suppose that X is a variety. We say that X is *diagonally split* if the diagonal D is compatibly Frobenius split in $X \times X$. Given an ample divisor on X, we say that X is *diagonally split along an ample effective divisor A* if there exists a Frobenius splitting $\phi : F_*\mathcal{O}_{X \times X} \to \mathcal{O}_{X \times X}$ that compatibly splits D and also factors through $F_*\mathcal{O}_{X \times X}(p_2^* A)$.

Proposition 2.2. *Suppose that X is a complete variety and suppose that $\mathscr{L}$ and $\mathscr{M}$ are line bundles on X. Consider the natural map*

$$m(\mathscr{L}, \mathscr{M}) : \Gamma(X, \mathscr{L}) \otimes \Gamma(X, \mathscr{M}) \to \Gamma(X, \mathscr{L} \otimes \mathscr{M}).$$

If either

(1) *$\mathscr{L}$ and $\mathscr{M}$ are ample and X is diagonally Frobenius split, or*

(2) *$\mathscr{L}$ and $\mathscr{M}$ are semi-ample (or simple nef?) and X is diagonally Frobenius split along an ample effective Cartier divisor A,*

then $m(\mathscr{L},\mathscr{M})$ is surjective.

Proof. We begin by recasting $m(\mathscr{L},\mathscr{M})$ as a different map. Now, $\Gamma(X,\mathscr{L})\otimes\Gamma(X,\mathscr{M})\cong\Gamma(X\times X, p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})$, and furthermore, if $i : D\subseteq X\times X$ is the inclusion map, then $i^*(p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})=\mathscr{L}\otimes\mathscr{M}$. Therefore, it is sufficient to show that the restriction map

$$\Gamma(X\times X, p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})\to\Gamma(D,(p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})|_D)$$

is surjective. In the first case, $p_1^*\mathscr{L}\otimes p_2^*\mathscr{M}$ is ample. Consider the following commutative diagram where ϕ is just the Frobenius splitting twisted by a line bundle:

$$\begin{array}{ccc} H^0(X\times X,\mathcal{O}_{X\times X}((p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})^{p^e})) & \xrightarrow{\phi} & H^0(X\times X,\mathcal{O}_{X\times X}(p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})) \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ H^0(D,\mathcal{O}_D((p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})^{p^e})) & \xrightarrow{\bar\phi} & H^0(D,\mathcal{O}_D(p_1^*\mathscr{L}\otimes p_2^*\mathscr{M})) \end{array}$$

By Serre vanishing, γ is surjective and $\bar\phi$ is also surjective because it is induced from a splitting. Thus δ is surjective as well and (1) is proven.

By composing Frobenius splittings along an both p_1^*A and p_2^*A, we obtain a Frobenius splitting along an ample divisor $B=p_1^*A^n\otimes p_2^*A^m$ on $X\times X$ for some integers $n,m>0$. Consider the restriction map

$$H^0(X\times X,\mathcal{O}_{X\times X}((p_1^*\mathscr{L}^r\otimes p_2^*\mathscr{M}^r)(B)))\to H^0(D,\mathcal{O}_D((p_1^*\mathscr{L}^r\otimes p_2^*\mathscr{M}^r)(B)))$$

for various integers r. The above argument shows that this map is surjective. Composing with the Frobenius splitting along B gives us a diagram

$$\begin{array}{ccc} H^0(X\times X,F^e_*\mathcal{O}_{X\times X}((p_1^*\mathscr{L}^r\otimes p_2^*\mathscr{M}^r)(B))) & \longrightarrow & H^0(D,F^e_*\mathcal{O}_D((p_1^*\mathscr{L}^r\otimes p_2^*\mathscr{M}^r)(B))) \\ \downarrow & & \downarrow \\ H^0(X\times X,\mathcal{O}_{X\times X}((p_1^*\mathscr{L}^r\otimes p_2^*\mathscr{M}^r))) & \longrightarrow & H^0(D,F^e_*\mathcal{O}_D((p_1^*\mathscr{L}^r\otimes p_2^*\mathscr{M}^r))) \end{array}$$

As before, the bottom row is surjective which completes the proof. □

Corollary 2.3. *Suppose that X is a diagonally Frobenius split projective variety, then every ample divisor is very ample and induces a projectively normal embedding. Furthermore, if it is diagonally Frobenius split along an ample divisor, then the algebra of sections of a semi-ample divisor is generated in degree 1. Furthermore, every semi-ample divisor is globally generated.*

Corollary 2.4. *Suppose in addition that X is Cohen-Macaulay and diagonally Frobenius split along an ample divisor, then X is arithmetically Cohen-Macaulay with respect to any ample line bundle.*

Proof. Choose A an ample effective divisor, this divisor is very ample and induces a projectively normal embedding by assumption. Thus we only have to show that $H^i(X,\mathcal{O}_X(vA))=0$ for $1\le i\le\dim(X)-1$ and all $v\in\mathbb{Z}$. But since A is ample, these vanishings hold via the usual Frobenius splitting arguments for $v \neq 0$. Consider $v = 0$, suppose that M is an ample divisor along which X is Frobenius split. We have

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X \to F^e_* \mathcal{O}_X(M)$$

splits and thus $\mathcal{O}_X \to F^{ne} \mathcal{O}_X(mM)$ where we can make $m \gg 0$. But then $H^i(X, \mathcal{O}_X) \to H^i(X, F^e_* \mathcal{O}_X(mM))$ splits, and the right side vanishes for $m \gg 0$. ☐

Remark 2.5. Various generalizations can be made to splittings of $X \times X \times \cdots \times X$. Furthermore, these can be used to prove that various section rings $R(X, \mathscr{L})$ are Koszul.

3. Toric varieties

In this section we briefly discuss Frobenius splittings on toric varieties. There are numerous good introductions to toric varieties available, the canonical reference is probably still [] although also see [].

Consider the torus $T \simeq (\mathbb{G}_m)^n = (\mathbb{A}^1 \setminus \{0\})^n$ where $k = \overline{k}$.

Definition 3.1. A *toric variety* is a normal variety X containing T as an open subset such that the natural action of T on itself by multiplication extends to an action on X.

Lemma 3.2. *A toric variety can be covered by Torus invariant affine open subsets. Each one of them is* $\operatorname{Spec} k[\mathbf{x}^{\lambda_1}, \ldots, \mathbf{x}^{\lambda_m}]$ *for some monomials* $\mathbf{x}^{\lambda_i}$.

Proof. We leave the first statement to the reader, as it is contained in any introductory text on toric varieties. For the second statement, notice that if $U = \operatorname{Spec} R$ is a torus invariant open affine subset, then if any polynomial $f = \sum a_i \mathbf{x}^i$ is in U, by using the torus action, it is clear that each monomial appearing in f is in R. The claimed statement follows. ☐

Now, the torus $T = \operatorname{Spec} k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ has a very natural Frobenius splitting $\Phi_c : F_* \mathcal{O}_T \to \mathcal{O}_T$, namely the one defined as follows:

$$\Phi_c(\mathbf{x}^\lambda) = \begin{cases} \mathbf{x}^{\lambda/p} & \text{if each entry in } \lambda \text{ is divisible by } p. \\ 0 & \text{otherwise.} \end{cases}$$

This is called the canonical Frobenius splitting (also see [BK05, Section 4]).

Proposition 3.3. *If* X *is a toric variety, then* Φ_c *induces a Frobenius splitting* $\Phi_c : F_* \mathcal{O}_X \to \mathcal{O}_X$.

Proof. We can work on an open affine set $U = \operatorname{Spec} R = \operatorname{Spec} k[\mathbf{x}^{\lambda_1}, \ldots, \mathbf{x}^{\lambda_m}]$. Since R is normal, if $\mathbf{x}^\lambda \in R$, then if $\lambda/p \in \mathbb{Z}^n$, we clearly see that $\mathbf{x}^{\lambda/p} \in R$ as well. Since $\Phi_c(1) = 1$, we have explicitly seen our Frobenius splitting. ☐

Definition 3.4. If X is a toric variety and $Z \subseteq X$ an irreducible subvariety, then we say that Z is a *torus invariant subvariety* if it is invariant under the torus action.

Example 3.5. In the toric variety $\mathbb{A}^2$, the torus invariant subvarieties are the two coordinate axes and also the origin.

Lemma 3.6. *Suppose that* X *is a toric variety, then* $Z \subseteq X$ *is a torus invariant subvariety if and only if* Z *is* Φ_c *compatible.*

Proof. We can assume that $X = \operatorname{Spec} k[\mathbf{x}^{\lambda_1}, \dots, \mathbf{x}^{\lambda_m}]$. A torus invariant subvariety thus corresponds to a prime ideal generated by monomials, and it is clear that any such ideal is Φ_c-compatible. Thus we suppose that Q is a Φ_c compatible ideal (note that Φ_c is surjective, so Q is automatically radical and $\Phi_c(F_* Q) = Q$). Further suppose that Q is prime. We will show that Q generated by monomials.

Suppose that $\sum a_i \mathbf{x}^{\lambda_i} = g \in Q$. We simply need to show that each $\mathbf{x}^{\lambda_i} \in Q$. Consider $h = \Phi_c^e(\mathbf{x}^{(p^e-1)\lambda_i} g)$. Clearly this polynomial contains $\mathbf{x}^{\lambda_i}$ as an entry. Now, consider $\Phi_c^e(\mathbf{x}^{(p^e-1)\lambda_i} \mathbf{x}^{\lambda_j}) = \mathbf{x}^{((p^e-1)\lambda_i + \lambda_j)/p^e}$. But

$$((p^e-1)\lambda_i + \lambda_j)/p^e = \lambda_i + \frac{\lambda_j - \lambda_i}{p^e}.$$

This is not in $\mathbb{Z}^n$ for $e \gg 0$ if $j \neq i$. Therefore, for $e \gg 0$, $\Phi_c^e(\mathbf{x}^{(p^e-1)\lambda_i} g) = \mathbf{x}^{\lambda_i}$ which proves that Q is generated by monomials. □

We now briefly review the theory of canonical divisors on toric varieties.

Lemma 3.7. *The anti-canonical divisor $-K_X$ in a toric variety X is equal to the sum of all the torus invariant divisors. It can also be identified with $X \setminus T$.*

Proof. See for example, []. □

Proposition 3.8. *The Frobenius splitting Φ_c above has associated divisor $-K_X$.*

Proof. Clearly the divisor $\Delta_{\Phi_c} \geq -K_X$ (since every torus invariant divisor is Φ_c-compatible). Therefore, we only have to observe that $\operatorname{Supp}(\Delta_{\Phi_c})$ is torus invariant.

However, on the torus T, Φ_c generates $\operatorname{Hom}_{\mathcal{O}_T}(F_* \mathcal{O}_T, \mathcal{O}_T)$ as an $\mathcal{O}_T$-module. □

Proposition 3.9. *Projective toric varieties in characteristic zero are log Fano and in characteristic $p > 0$ are globally F-regular.*

Proof. Suppose that X is a projective toric variety in characteristic zero. Suppose that A is an ample effective torus invariant divisor (it is a general fact that $\operatorname{Supp}(A) = \operatorname{Supp}(-K_X)$. Choose rational $\varepsilon > 0$ such that $\Delta := -K_X - \varepsilon A > 0$. Choose a toric log resolution $\pi : \widetilde{X} \to X$ of (X, Δ), we know that

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) = K_{\widetilde{X}} - \pi^*(K_X - K_X - \varepsilon A) = K_{\widetilde{X}} + \varepsilon \pi^* A.$$

It is clear that $\operatorname{Supp}(\pi^* A) = \operatorname{Supp}(-K_{\widetilde{X}})$, thus all the coefficients of $K_{\widetilde{X}} + \varepsilon \pi^* A$ are strictly bigger than -1 (which are the coefficients of $K_{\widetilde{X}}$).

Now suppose that X is a projective toric variety in characteristic $p > 0$. Again choose A an ample effective torus invariant divisor. The section ring $S(A)$ is always a monomial algebra, and therefore it is always strongly F-regular (since it is a summand of $k[x_1, \dots, x_n]$). The fact that the section ring is strongly F-regular is then easily seen to imply that X is globally F-regular. □

An important open question in the study of toric varieties is the following (which I have seen attributed to Oda):

Question 3.10. Suppose that X is a smooth toric variety and $\mathscr{L}$ is an ample line bundle. Then does $\mathscr{L}$ induce a projectively normal embedding into some projective space? It is known that $\mathscr{L}$ is always very ample.

Several years ago, it was hoped that the Frobenius splitting methods including Frobenius splitting along diagonals, would be enough to prove this result. The most immediate problem is that the diagonal in $X \times X$ is not torus invariant, and therefore it is *NOT* compatibly Frobenius split by Φ_c (this caused some confusion in the past). However Sam Payne has analyzed exactly when there exists a Frobenius splitting of $X \times X$ which compatibly splits the diagonal (it's just not the toric one).

If Δ is a fan in a lattice N and M is the dual lattice, Payne defined

$$\mathbf{F}_X := \{u \in M | -1 \leq \langle u, v_\rho \rangle \leq 1 \text{ where } v_\rho \text{ is a primitive generator of a ray in } \Delta.\}$$

Theorem 3.11. [Pay09] *A toric variety $X = X(\Delta)$ is diagonally Frobenius split if and only if the interior of $\mathbf{F}_X$ contains the interior of every equivalence class of $(\frac{1}{p}M)/M$.*

References

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