F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 11/9-2010

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Before continuing on, we need a very brief introduction to Matlis/local-duality. Suppose that $(R, \mathfrak{m})$ is a local ring. We know every R-module lives inside an injective R-module. In particular, $R/\mathfrak{m}$ lives inside an injective R-module I. It turns out that there is in some sense a smallest (up to containment) injective module E containing $R/\mathfrak{m}$. This module is unique up to isomorphism and is called the injective hull of $R/\mathfrak{m}$ and will be denoted by $E = E_{R/\mathfrak{m}}$.

Theorem 0.1 (Matlis)**.** [BH93] $\operatorname{Hom}_R(\_\_, E)$ *is an exact functor which (faithfully) takes finitely generated R-modules to artinian R-module. Furthermore, if R is complete, then the functor (faithfully) takes artinian R-modules to finitely generated R-modules, induces an equivalence of categories between the two sets, and applying it twice is an isomorphism. Finally,* $\operatorname{Hom}_R(\_\_, E)$ *always induces an equivalence of the category of finite length R-modules (i.e. , Noetherian + Artinian modules).*

Theorem 0.2 (Grothendieck)**.** [Har66] *With notation as above,* $\operatorname{Hom}_R(h^{-i}(\omega_R^{\bullet}), E) \cong h^{-i}(\mathbf{R}\operatorname{Hom}_R(\omega_R^{\bullet}, E))$ $H^i_{\mathfrak{m}}(R)$*. More generally for $M \in D^b_{\mathrm{coh}}(R)$ there is a functorial isomorphism*

$$\operatorname{Hom}_R(h^{-j}\mathbf{R}\operatorname{Hom}_R(M, \omega_R^{\bullet}), E) \cong \mathbb{H}^j_{\mathfrak{m}}(M).$$

Corollary 0.3. *With notation as above,* $\operatorname{Hom}_R(F^e_* \omega_R, E) \cong H^d_{\mathfrak{m}}(F^e_* R)$.

Corollary 0.4. *An F-finite ring R is F-rational if and only if it is:*

(a) *R is Cohen-Macaulay, and*
(b) *for every finite extension $R \subseteq S$, the natural map $T : \omega_S \to \omega_R$ is surjective.*

Condition (b) and also be replaced by

(b*) *for every generically finite proper map $\pi : Y \to \operatorname{Spec} R$, the natural map $T : \pi_* \omega_Y \to \omega_R$ is surjective, or*
(b**) *for every alteration $\pi : Y \to \operatorname{Spec} R$, the natural map $T : \pi_* \omega_Y \to \omega_R$ is surjective.*

Proof. It is harmless to assume that R is normal (otherwise the normalization map breaks condition (b), (b*) and (b**)).

First we will show that F-rational implies (b*) (which obviously implies (b) and (b**)). But this is easy, simply consider the commutative diagram

$$\begin{array}{ccc} F_* \pi_* \omega_Y & \xrightarrow{\pi_* \Psi_Y} & \pi_* \omega_Y \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ F_* \omega_R & \xrightarrow[\Psi_R]{} & \omega_R \end{array}$$

The image of T is clearly Ψ_R-stable and non-zero, and F-rational implies that there are no proper Ψ_R-stable submodules.

Conversely, suppose we have conditions (a) and (b) (note that condition (b) is automatically implied by (b*) and (b**)). Suppose that R is not F-rational. By localizing at the generic point of the non-F-rational locus, we may assume that $(R, \mathfrak{m})$ is a local d-dimensional ring which is F-rational on the punctured spectrum. This means that $\omega_R/\tau(\omega_R)$ is supported at the maximal ideal. We set E to be an injective hull of $R/\mathfrak{m}$ and apply $\operatorname{Hom}_R(\_\_, E)$ to the short exact sequence:

$$0 \to \tau(\omega_R) \to \omega_R \to \omega_R/\tau(\omega_R) \to 0$$

yielding

$$0 \leftarrow \tau(\omega_R)^\vee \leftarrow H^d_\mathfrak{m}(R) \leftarrow (\omega_R/\tau(\omega_R))^\vee \leftarrow 0.$$

We knew that $\tau(\omega_R)$ is $\Phi_R : F_*R \to R$ stable. It follows that its dual is stable under the Frobenius action $H^d_\mathfrak{m}(R) \to H^d_\mathfrak{m}(F^e_* R)$. Its dual is a finite length Frobenius stable submodule, thus there exists a finite extension $R \subseteq S$ such that $(\omega_R/\tau(\omega_R))^\vee$ is sent to zero in $H^d_\mathfrak{m}(S)$. Consider the diagram:

$$\omega_S \to \omega_R \to \omega_R/\operatorname{Image}(\omega_S) \to 0$$

The dual is

$$H^d_\mathfrak{m}(S) \leftarrow H^d_\mathfrak{m}(R) \leftarrow K \leftarrow 0.$$

We know that $(\omega_R/\tau(\omega_R))^\vee$ is contained in K. Thus $\tau(\omega_R) = \operatorname{Image}(\omega_S \to \omega_R)$. □

Remark 0.5. The submodule $(\omega_R/\tau(\omega_R))^\vee \subseteq H^d_\mathfrak{m}(R)$ is often denoted by $0^*_{H^d(R)}$ and is called the *tight closure of zero in* $H^d_\mathfrak{m}(R)$.

The proof leads us to the following question. Does there always exist a finite map $R \subseteq S$ such that $\tau(R) = \operatorname{Image}(\omega_S \to \omega_R)$?

Theorem 0.6. [HL07]*[cf Hochster-Yao] Suppose R is an F-finite domain. Then there always exists a finite map $R \subseteq S$ such that $\tau(R) = \operatorname{Image}(\omega_S \to \omega_R)$ and therefore*

$$\tau(\omega_R) = \bigcap_{R \subseteq S} \operatorname{Image}(\omega_S \to \omega_R).$$

More generally,

$$\tau(\omega_R) = \bigcap_{f : Y \to \operatorname{Spec} R \text{ a regular alteration}} \operatorname{Image}(f_*\omega_Y \to \omega_R).$$

Proof. The statement is local so we assume that R is a local ring with maximal ideal $\mathfrak{m}$.

First we show that the second statement follows from the first. To do this, we simply observe that $\tau(\omega_R) \subseteq \operatorname{Image}(f_*\omega_Y \to \omega_R)$ for any generically finite proper dominant map $f : Y \to \operatorname{Spec} R$ (this is based on the usual argument used to prove "F-rational ($\Rightarrow$) rational", which is essentially due to K. Smith, [Smi97]).

To see this, consider the diagram

$$\begin{array}{ccc} Y & \xrightarrow{F} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ \operatorname{Spec} R & \xrightarrow{F} & \operatorname{Spec} R \end{array}$$

where the horizontal arrows are Frobenius. It is an easy application of Grothendieck duality that we have a commutative diagram:

$$\begin{array}{ccc} F_*f_*\omega_Y & \xrightarrow{f_*\Phi_Y} & f_*\omega_Y \\ \downarrow & & \downarrow \\ F_*\omega_R & \xrightarrow{\Phi_R} & \omega_R \end{array}$$

The image of $f_*\omega_Y \to \omega_R$ is non-zero at every maximal dimensional component of $\operatorname{Spec} R$ since f is generically finite and dominant. From this diagram and the definition of the parameter test submodule it immediately follows that $\operatorname{Image}(f_*\omega_Y \to \omega_R)$ contains $\tau(\omega_R)$.

It hence remains to show that we can find some finite map where the containment is indeed equality. For this we closely follow the strategy of [HL07]: Choose $\eta \in \operatorname{Spec} R$ to be a generic point of the non-F-rational locus of R. We know that $(\omega_R/\tau(\omega_R))^\vee = 0^*_{\mathbb{H}^d_{\mathfrak{m}}(R)}$ (where $(\cdot)^\vee$ denotes the Matlis dual by []. Because the punctured spectrum of R is F-rational, $\omega_R/\tau(\omega_R)$ and thus also $0^*_{\mathbb{H}^d_{\mathfrak{m}}}(R)$ has finite length.

It follows from the equational lemma above, that there exists a finite extension of reduced rings $R_\eta \subseteq S_\eta$ such that the image of $0^*_{\mathbb{H}^{\dim R_\eta}_\eta(R_\eta)}$ in $H^{\dim R_\eta}_\eta(S_\eta)$ is zero. By taking the normalization of R inside the total field of fractions of S_η, we may assume that S_η is indeed the localization of some S' at η. Because $0^*_{\mathbb{H}^{\dim R_\eta}_\eta(R_\eta)} \to H^{\dim R_\eta}_\eta(S_\eta)$ is zero, the Matlis dual map $(\omega_S)_\eta \to (\omega_R/\tau(\omega_R))_\eta$ is also zero.

By doing this for each generic point of the non-F-rational locus of R and taking a common extension, we can set S^1 to be a common extension of all the S'. We thus have $I_1 := \operatorname{Image}(\omega_{S^1} \to \omega_R)$ such that the support of $I_1/\tau(\omega_R)$ is of strictly smaller dimension than the non-F-rational locus. Choose $\eta_1 \in \operatorname{Spec} S^1$, a generic point of that support and suppose that $\dim R_{\eta_1} = d_1$. Therefore, $(I_1/\tau(\omega_R))_{\eta_1}$ has finite length.

Consider the map $g_1 : H^{d_1}_{\eta_1}(R_{\eta_1}) \to H^{d_1}_{\eta_1}(S^1_{\eta_1})$ and note that the image of $\omega_{S^1})_{\eta_1} \to \omega_R/\tau(\omega_R)$ has support at η_1 which implies that the image of $g_1(0^*_{H^{d_1}_{\eta_1}(R_{eta_1})}) \subseteq H^{d_1}_{\eta_1}(S^1_{\eta_1})$ is also finite length.

Choose z in that image. We know $z, z^p, z^{p^2}, \ldots$ are also contained in $g_1(0^*_{H^{d_1}(R_{eta_1})})$ since $0^*_{H^{d_1}_{\eta_1}(R_{\eta_1})}$ is stable under the Frobenius action. Therefore, there exists a finite extension $S^1_{\eta_1} \subseteq S''^1_{\eta_1}$ such that z is sent to zero under the map $H^{d_1}_{\eta_1}(S^1_{\eta_1}) \to H^{d_1}_{\eta_1}(S''^1_{\eta_1})$. Because $(\tau(\omega_{R_{\eta_1}}))^\vee/\ker(g_1)$ is finite length, we can find a common extension $S_{\eta_1} \subseteq S'^1_{\eta_1}$ which kills $0^*_{H^{d_1}_{\eta_1}(R_{\eta_1})}$. Thus the map $\omega_{S'_{\eta_1}} \to \omega_{R_{\eta_1}}$ has image $\tau(\omega_{R_{\eta_1}})$. Set S^2 to be the normalization of R inside the fraction field of S'^1. Define $I_2 := \operatorname{Image}(\omega_{S^2} \to \omega_R)$.

It follows that $(I_2/\tau(\omega_R))$ has support a strictly smaller closed subset than $(I_1/\tau(\omega_R))$. Continuing in this way, Noetherian induction tells us that eventually $\tau(\omega_R)$ is the image of some map $\omega_{S^n} \to \omega_R$. $\square$

References

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