F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 11/30-2010

KARL SCHWEDE

1. Globally F-regular varieties

Definition 1.1. Let (X, Δ) be a pair, where X is a normal irreducible F-finite scheme of prime characteristic p and Δ is an effective $\mathbb{Q}$ divisor on X. The pair (X, Δ) is *globally F-regular* if, for every effective divisor D, there exists some $e > 0$ such that the natural map $\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D)$ splits (in the category of $\mathcal{O}_X$-modules).

X itself is called *globally F-regular* if $(X, 0)$ is globally F-regular.

Lemma 1.2. *If (X, Δ) is globally F-regular, then (X, Δ') is globally F-regular for any $\Delta' \leq \Delta$. The corresponding statement for globally sharply F-split pairs also holds.*

Proof. This follows easily from the following simple observation: If a map of coherent sheaves $\mathscr{L} \xrightarrow{g} \mathscr{F}$ on a scheme X splits, then there is also a splitting for any map $\mathscr{L} \xrightarrow{h} \mathscr{M}$ through which g factors. Indeed, factor g as $\mathscr{L} \xrightarrow{h} \mathscr{M} \xrightarrow{h'} \mathscr{F}$. Then if $s : \mathscr{F} \to \mathscr{L}$ splits g, it is clear that the composition $s \circ h'$ splits h. Now we simply observe that if $\Delta' \leq \Delta$, we have a factorization

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta' \rceil + D) \hookrightarrow F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta \rceil + D),$$

so the result follows. $\square$

Remark 1.3. On an affine variety, Globally F-regular is the same as strongly F-regular (one can certainly take $D = \operatorname{div}(c)$ for various $c \in \mathcal{O}_X$, and every effective divisor D is less than or equal to a Cartier divisor). However, since every globally F-regular variety is clearly F-split, not every (locally) strongly F-regular variety is globally F-regular.

We now establish a useful criterion for global F-regularity, generalizing well-known results for the local case [HH89, Theorem 3.3] and the "boundary-free" case [Smi00, Theorem 3.10].

Theorem 1.4. *The pair (X, Δ) is globally F-regular if (and only if) there exists some effective (usually ample) divisor C on X satisfying the following two properties:*

(i) *There exists an $e > 0$ such that the natural map*

$$\mathcal{O}_X \to F^e_* \mathcal{O}_X(\lceil (p^e - 1)\Delta + C \rceil)$$

splits.

(ii) *The pair $(X \setminus C, \Delta|_{X \setminus C})$ globally F-regular (for example, affine and locally F-regular).*

Proof of Theorem 1.4. Let X_C denote the open set complimentary to C. Now fix any effective divisor C' on X. By hypothesis (ii), we can find e' and an $\mathcal{O}_X$-module homomorphism

$\phi : F^{e'}_* \mathcal{O}_{X_C}(\lceil (p^{e'}-1)\Delta|_{X_C} + C'|_{X_C}\rceil) \to \mathcal{O}_{X_C}$ that sends 1 to 1. In other words, ϕ is a section of the reflexive sheaf

$$\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^{e'}_* \mathcal{O}_X(\lceil (p^{e'}-1)\Delta + C'\rceil), \mathcal{O}_X)$$

over the open set X_C. Thus on the non-singular locus U of X (really, we need the Cartier locus of C), we can choose $m_0 > 0$ so that $\phi|_U$ is the restriction of a global section ϕ_m of

$$\begin{aligned} &\mathscr{H}\mathrm{om}_{\mathcal{O}_U}(F^{e'}_* \mathcal{O}_U(\lceil (p^{e'}-1)\Delta + C'\rceil), \mathcal{O}_U) \otimes \mathcal{O}_U(mC) \\ &\cong \mathscr{H}\mathrm{om}_{\mathcal{O}_U}(F^{e'}_* \mathcal{O}_U(\lceil (p^{e'}-1)\Delta + C'\rceil), \mathcal{O}_U(mC)) \end{aligned} \tag{1}$$

over U, for all $m \geq m_0$; see [Har77, Chapter II, Lemma 5.14(b)]. Note that ϕ_m still sends 1 to 1. Now, since the involved sheaves are reflexive, this section extends uniquely to a global section of $\mathscr{H}\mathrm{om}_{\mathcal{O}_X}(F^{e'}_* \mathcal{O}_X(\lceil (p^{e'}-1)\Delta + C'\rceil), \mathcal{O}_X(mC))$, also denoted ϕ_m over the whole of X.

Consider an m of the form $m = p^{(n-1)e} + \dots p^e + 1$, where e is the number guaranteed by hypothesis (i). Tensoring the map ϕ_m from Equation (1) with $\mathcal{O}_X(\lceil (p^{ne}-1)\Delta\rceil)$, we have an induced map

$$F^{e'}_* \mathcal{O}_X(\lceil (p^{e'}-1)\Delta\rceil + C' + p^{e'}\lceil (p^{ne}-1)\Delta\rceil) \to \mathcal{O}_X(\lceil (p^{ne}-1)\Delta + mC\rceil).$$

Now, as in Lemma 1.2, it follows that there is a map

$$\psi : F^{e'}_* \mathcal{O}_X(\lceil (p^{ne+e'}-1)\Delta + C'\rceil) \to \mathcal{O}_X(\lceil (p^{ne}-1)\Delta + mC\rceil)$$

which sends 1 to 1.

By composing the splitting from hypothesis (i) with itself $(n-1)$-times and after twisting appropriately (compare with [Tak04, Proof of Lemma 2.5] and [Sch09]), we obtain a map

$$\theta : F^{ne}_* \mathcal{O}_X(\lceil (p^{ne}-1)\Delta + (p^{(n-1)e} + \cdots + p^e + 1)C\rceil) = F^{ne}_* \mathcal{O}_X(\lceil (p^{ne}-1)\Delta + mC\rceil) \to \mathcal{O}_X$$

which sends 1 to 1.

Combining the maps θ and ψ, we obtain a composition

$$F^{ne+e'}_* \mathcal{O}_X(\lceil (p^{ne+e'}-1)\Delta + C'\rceil) \xrightarrow{F^{ne}_*(\psi_\Delta)} F^{ne}_* \mathcal{O}_X(\lceil (p^{ne}-1)\Delta + mC\rceil) \xrightarrow{\theta} \mathcal{O}_X$$

which sends 1 to 1 as desired. The proof is complete. □

Theorem 1.5. *Let X be a normal scheme quasiprojective over an F-finite local ring with a dualizing complex and suppose that B is an effective $\mathbb{Q}$-divisor on X.*

(i) *If the pair (X, B) is globally F-regular, then there is an effective $\mathbb{Q}$-divisor Δ such that $(X, B + \Delta)$ is globally F-regular with $K_X + B + \Delta$ anti-ample.*
(ii) *Similarly, if (X, B) is globally sharply F-split, then there exists an effective $\mathbb{Q}$-divisor Δ such that $(X, B + \Delta)$ is globally sharply F-split with $K_X + B + \Delta$ $\mathbb{Q}$-trivial.*

In both (i) and (ii), the denominators of the coefficients of $B+\Delta$ can be assumed not divisible by the characteristic p.

Proof of Theorem 1.5. First, without loss of generality, we may assume that the $\mathbb{Q}$-divisor B has no denominators divisible by p, we won't prove this here but it is straightforward.

We first prove statement (ii), which follows quite easily. Suppose that (X, B) is globally sharply F-split. Consider a splitting

$$\mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X \longrightarrow F^e_* \mathcal{O}_X((p^e-1)B) \xrightarrow{\phi} \mathcal{O}_X$$

where $(p^e-1)B$ is an integral divisor. Apply $\mathscr{H}om_{\mathcal{O}_X}(\_\_,\mathcal{O}_X)$ to this splitting. We then obtain the following splitting,

$$\mathcal{O}_X \longleftarrow F^e_*\mathcal{O}_X((1-p^e)K_X) \longleftarrow F^e_*\mathcal{O}_X((1-p^e)(K_X+B)) \xleftarrow{\phi^\vee} \mathcal{O}_X.$$

The image of 1 under $\phi^\vee$ determines a divisor D' which is linearly equivalent to $(1-p^e)(K_X+B)$. This produces a composition

$$\mathcal{O}_X \longleftarrow F^e_*\mathcal{O}_X(D'+(p^e-1)B) \longleftarrow F^e_*\mathcal{O}_X(D') \xleftarrow{\phi^\vee} \mathcal{O}_X. \tag{2}$$

Set $\Delta_1 = \frac{1}{p^e-1}D'$. Then the pair $(X, B+\Delta_1)$ is globally sharply F-split with the splitting given by Equation (2). But also, it is log Calabi Yau, since

$$K_X + B + \Delta_1 \sim_{\mathbb{Q}} K_X + B + \frac{1}{p^e-1}(1-p^e)(K_X+B) = 0.$$

This completes the proof of (ii).

More work is required to prove (i). Suppose that (X, B) is globally F-regular. Then it is also globally sharply F-split, and we may pick Δ_1 as in (ii). Choose H to be a very ample effective divisor such that $\operatorname{Supp}\Delta_1 \subseteq \operatorname{Supp} H$. Consider a splitting

$$\mathcal{O}_X \longrightarrow F^f_*\mathcal{O}_X(H) \longrightarrow F^f_*\mathcal{O}_X((p^f-1)B+H) \xrightarrow{\psi} \mathcal{O}_X,$$

such that $(p^f-1)B$ is integral. Apply $\mathscr{H}om_{\mathcal{O}_X}(\_\_,\mathcal{O}_X)$ to this splitting to obtain a dual splitting,

$$\mathcal{O}_X \longleftarrow F^f_*\mathcal{O}_X((1-p^f)K_X - H) \longleftarrow F^f_*\mathcal{O}_X((1-p^f)(K_X+B)-H) \xleftarrow{\psi^\vee} \mathcal{O}_X \tag{3}$$

The image of 1 under $\psi^\vee$ determines a divisor D'' which is linearly equivalent to $(1-p^f)(K_X+B)-H$. Set $\Delta_2 = \frac{1}{p^f-1}D''$. Note that

$$K_X + B + \Delta_2 \sim_{\mathbb{Q}} \frac{-1}{p^f-1}H$$

which is anti-ample. Also note that the splitting in line (3) demonstrates the pair $(X, B+\Delta_2)$ to be globally sharply F-split. Even better, line (3) also demonstrates $(X, B+\Delta_2+\frac{1}{p^f-1}H)$ to be globally sharply F-split.

We now make use of Lemma 1.6 below to complete the proof. In addition to the globally F-regular pair (X, B), we have constructed divisors Δ_1 and Δ_2 satisfying

(i) $(X, B+\Delta_1)$ is globally sharply F-split with $K_X+B+\Delta_1 \sim_{\mathbb{Q}} 0$; and
(ii) $(X, B+\Delta_2)$ is globally sharply F-split with $K_X+B+\Delta_2$ anti-ample.
(iii) $(X, B+\Delta_2+\delta H)$ is globally sharply F-split for some small positive δ.

Now we apply Lemma 1.6(i) to the divisors described in (i) and (iii) above. We thus fix positive rational numbers ϵ_1, ϵ_2, with $\epsilon_1+\epsilon_2 = 1$ such that

$$(X, \epsilon_1(B+\Delta_1)+\epsilon_2(B+\Delta_2+\delta H)) = (X, B+\epsilon_2\Delta_2+\epsilon_1\Delta_1+\epsilon_2\delta H)$$

is globally sharply F-split. Since the support of Δ_1 is contained in the support of H, it follows from Lemma 1.2 that

$$(X, B+\epsilon_2\Delta_2+(\epsilon_1+\epsilon')\Delta_1) \tag{4}$$

is globally sharply F-split for some small positive ϵ'. But also $(X, B + \epsilon_2\Delta_2)$ is globally F-regular, as one sees by applying Lemma 1.6(iii) to the globally F-regular pair (X, B) and the globally sharply F-split pair $(X, B + \Delta_2)$.

Finally, another application of Lemma 1.6(iii), this time to the globally F-regular pair $(X, B + \epsilon_2\Delta_2)$ and the globally sharply F-split pair of line (4), implies that $(X, B + \epsilon_2\Delta_2 + \epsilon_1\Delta_1)$ is globally F-regular. Set $\Delta = \epsilon_1\Delta_1 + \epsilon_2\Delta_2$. We conclude that the pair $(X, B + \Delta)$ is globally F-regular, and

$$K_X + B + \Delta = \epsilon_1(K_X + B + \Delta_1) + \epsilon_2(K_X + B + \Delta_2)$$

is anti-ample (from (i) and (ii) just above). This completes the proof of (i) and hence Theorem 1.5. □

Lemma 1.6. *Consider two pairs (X, B) and (X, D) on a normal F-finite scheme X.*

- (i) *If both pairs are globally sharply F-split, then there exist positive rational numbers ϵ arbitrarily close to zero such that the pair $(X, \epsilon B + (1 - \epsilon)D)$ is globally sharply F-split.*
- (ii) *If (X, B) is globally F-regular and (X, D) is globally sharply F-split, then there exist positive rational numbers ϵ arbitrarily close to zero such that the pair $(X, \epsilon B+(1-\epsilon)D)$ is globally F-regular.*
- (iii) *In particular, if (X, B) is globally F-regular and $(X, B+\Delta)$ is globally sharply F-split, then $(X, B + \delta\Delta)$ is globally F-regular for all rational $0 < \delta < 1$.*

In (i) and (ii), the number ϵ can be assumed to have denominator not divisible by p.

Proof of Lemma 1.6. First note that (iii) follows from (ii) by taking D to be $(B + \Delta)$. Since $(1 - \epsilon)$ can be taken to be arbitrarily close to 1, we can choose it to exceed any given $\delta < 1$. Hence, the pair $(X, B + \delta\Delta)$ is globally F-regular for all positive $\delta < 1$, by Lemma 1.2.

For (i), we prove that we can take ϵ to be any rational number of the form

$$\epsilon = \frac{p^e - 1}{p^{(e+f)} - 1} \tag{5}$$

where e and f are sufficiently large and divisible (but independent) integers. Take e large and divisible enough so there exists a map $\phi : F^e_*\mathcal{O}_X(\lceil (p^e - 1)B\rceil) \to \mathcal{O}_X$ which splits the map $\mathcal{O}_X \to F^e_*\mathcal{O}_X(\lceil (p^e - 1)B\rceil)$. Likewise, take f large and divisible enough so there exists a map $\psi : F^f_*\mathcal{O}_X(\lceil (p^f - 1)D\rceil) \to \mathcal{O}_X$ which splits the map $\mathcal{O}_X \to F^f_*\mathcal{O}_X(\lceil (p^f - 1)D\rceil)$.

Consider the splitting

$$\mathcal{O}_X \longrightarrow F^e_*\mathcal{O}_X(\lceil (p^e - 1)B\rceil) \xrightarrow{\phi} \mathcal{O}_X.$$

Because all the sheaves above are reflexive and X is normal, we can tensor with $\mathcal{O}_X(\lceil (p^f - 1)D\rceil$ to obtain a splitting

$$\mathcal{O}_X(\lceil (p^f - 1)D\rceil) \longrightarrow F^e_*\mathcal{O}_X(\lceil (p^e - 1)B\rceil + p^e\lceil (p^f - 1)D\rceil) \longrightarrow \mathcal{O}_X(\lceil (p^f - 1)D\rceil).$$

Applying F^f_* to this splitting, and then composing with ψ we obtain the following splitting,

$$\mathcal{O}_X \longrightarrow F^{e+f}_*\mathcal{O}_X(\lceil (p^e - 1)B\rceil + p^e\lceil (p^f - 1)D\rceil) \longrightarrow \mathcal{O}_X$$

However, we also note that

$$\lceil (p^e - 1)B\rceil + p^e\lceil (p^f - 1)D\rceil \geq \lceil (p^e - 1)B + p^e(p^f - 1)D\rceil$$

which implies that we also have a splitting

$$\mathcal{O}_X \longrightarrow F_*^{e+f}\mathcal{O}_X(\lceil (p^e-1)B + p^e(p^f-1)D\rceil) \longrightarrow \mathcal{O}_X$$

If we then multiply $(p^e-1)B + p^e(p^f-1)D$ by $\frac{1}{p^{(e+f)}-1}$, the proof of (i) is complete for the choice of ϵ given in line 5.

Now, to prove (ii), we use Theorem 1.4. Choose an effective integral divisor C whose support contains the support of D and such that the pair $(X \setminus C, D|_{X\setminus C})$ is globally F-regular. Since there exists a splitting of

$$\mathcal{O}_X \to F_*^f\mathcal{O}_X(\lceil (p^f-1)B + C\rceil),$$

it follows that the pair $(X, B + \frac{1}{p^f-1}C)$ is globally sharply F-split. Applying part (i) of the Lemma to the pairs $(X, B + \frac{1}{p^f-1}C)$ and (X, D), we conclude that

$$(X, \epsilon(B + \frac{1}{p^f-1}C) + (1-\epsilon)D)$$

is globally sharply F-split. Re-writing, we have

$$(X, \epsilon B + (1-\epsilon)D + \epsilon' C)$$

is globally sharply F-split for ϵ and ϵ' arbitrarily close to zero.

We now apply Theorem 1.4 to the pair $(X, \Delta) = (X, \epsilon B + (1-\epsilon)D)$. Restricted to $X \setminus C$, this pair is globally F-regular, and we've just shown that for sufficiently small ϵ', the pair $(X, \Delta + \epsilon' C)$ is globally sharply F-split. Using Lemma 1.4 we conclude that (X, Δ) is globally F-regular.

Finally, note that because of the explicit choice of ϵ in line (3), it is clear its denominator can be assumed not divisible by p. □

Corollary 1.7. *If X is globally F-regular, then X there exists a divisor $\Delta \geq 0$ such that (X, Δ) is log Fano.*

Straightforward techniques involving cones imply the following converse.

Theorem 1.8. *Let X be a normal projective variety over a field of characteristic zero. If (X, Δ) is a Kawamata log terminal pair such that $K_X + \Delta$ is anti-ample (ie, (X, Δ) is log Fano), then (X, Δ) has globally F-regular type.*

Proof. The idea of the proof is the following lemma. X in characteristic $p > 0$ is globally F-regular if and only if the section ring with respect to an ample divisor is strongly F-regular. Also, for X in characteristic zero, (X, Δ) is log Fano if and only if the section ring pair (S, Δ_S), associated to an ample divisor, is Kawamata log terminal. Now reduce to characteristic $p \gg 0$. □

Theorem 1.9. *Let X be a normal projective variety over a field of prime characteristic. Let L be a Cartier divisor on X such that $L \sim_{\mathbb{Q}} M + \Delta$, where M is a nef and big $\mathbb{Q}$-divisor and the pair (X, Δ) is globally F-regular. Then $H^i(X, \mathcal{O}_X(-L)) = 0$ for $i < \dim X$.*

Proof. Because L is big, we can fix $f \gg 0$ so that there exists an effective E linearly equivalent to $p^f L$. By taking f larger if necessary, we can also assume that for all large and sufficiently divisible e,

(1) $p^f(p^e-1)\Delta$ and $p^f(p^e-1)M$ are integral,

(2) $\mathcal{O}_X(p^f(p^e-1)L) \cong \mathcal{O}_X(p^f(p^e-1)(M+\Delta))$.

Since M is nef and big, there exists an effective divisor D such that $nM - D$ is ample for all $n \gg 0$; see [Laz04, Cor 2.2.7]. Because (X, Δ) is globally F-regular, for all sufficiently large integers g, the map

$$\mathcal{O}_X \to F^g_* \mathcal{O}_X(\lceil (p^g-1)\Delta \rceil + D + E)$$

splits. By choosing g large enough, we may assume that $g = f + e$ where f is the fixed integer above and $e > 0$ is such that both (1) and (2) are satisfied above. Also, we can assume that $p^f(p^e-1)M - D$ is ample. Therefore, the map

$$\mathcal{O}_X \to F^{e+f}_* \mathcal{O}_X(p^f(p^e-1)\Delta + D + E)$$

splits since $p^f(p^e-1)\Delta \leq \lceil (p^{e+f}-1)\Delta \rceil$. Tensoring (on the smooth locus, and extending as usual) with $\mathcal{O}_X(-L)$ and taking cohomology, we have a splitting of the map

$$H^i(X, \mathcal{O}_X(-L)) \to H^i(X, F^{e+f}_* \mathcal{O}_X(-p^{e+f}L + p^f(p^e-1)\Delta + D + E)).$$

In particular, this map on cohomology is injective for all sufficiently large and divisible e.

However,

$$\begin{aligned} -p^{e+f}L + p^f(p^e-1)\Delta + D + E = \\ -(p^{e+f} - p^f)L - p^f L + p^f(p^e-1)\Delta + D + E \sim \\ (-p^f(p^e-1)M - p^f(p^e-1)\Delta) + p^f(p^e-1)\Delta + D + \big(E - p^f L\big) \sim \\ -p^f(p^e-1)M + D \end{aligned}$$

which is anti-ample. Therefore, $H^i(X, \mathcal{O}_X(-p^{e+f}L + p^f(p^e-1)\Delta + D + E))$ vanishes for $i < \dim X$ since X is globally F-regular, by [Smi00, Corollary 4.4], see also [BK05]. Because of the injection above, it follows that $H^i(X, \mathcal{O}_X(-L))$ vanishes, and the proof is complete. $\square$

2. Criteria for F-splitting of varieties

In the past, we've see Fedder's criteria for Frobenius splitting of algebraic varieties. Now, suppose that X is a variety over an algebraically closed field of characteristic $p > 0$. We will discuss the Mehta-Ramanathan criterion of Frobenius splitting, which is very useful in practice.

We've recently discussed using Cartier-operator as a way to construct explicitly the dual of Frobenius, $F_*\omega_X \to \omega_X$. Recall this was constructed as follows: we have the isomorphism $C^{-1} : \Omega^i_X(\log E) \cong \mathcal{H}^i(F_*(\Omega^{\bullet}_X(\log E))$. Take $E = 0$ and $i = d = \dim X$, this give us $\omega_X \cong \mathcal{H}^d(F_*\Omega^{\bullet}_X)$. But for $i > d$, the terms $F_*\Omega^i_X$ of the complex $F_*\Omega^{\bullet}_X$ are zero, and so we have a surjection $F_*\omega_X \to \omega_X$. This can be identified with the canonical dual of Frobenius.

Lemma 2.1. [BK05, Lemma 1.3.6] *Suppose that $x \in X$ is a smooth point of an n-dimensional variety X over an algebraically closed field k. Then the map $T : F_*\omega_X \to \omega_X$ is described by the following formula. For any set of generators $t_1, \dots, t_n$ of the maximal ideal of $\mathcal{O}_{X,x}$*

$$T(f dt_1 \wedge \dots \wedge dt_n) = S(f) dt_1 \wedge \dots \wedge dt_n$$

where S is defined on $k[[t_1, \dots, t_n]] \supseteq \mathcal{O}_{X,x}$ as the map which sends the monomial $t_1^{p-1} \dots t_n^{p-1}$ to 1 and the other monomials to zero.

This proof is taken from [BK05]. Certainly $dt_1 \wedge \dots dt_n$ generates ω_X as an $\mathcal{O}_X$-module as well, which identifies $\omega_{X,x}$ with $\mathcal{O}_{X,x}$. The completion of $\omega_X / d(\Omega_X^{d-1})$ is thus identified with $k[[x_1, \dots, x_n]]/J$ where J is the *vector-space* spanned by all partial derivatives of $h \in k[[x_1, \dots, x_n]$. To see this, simply note that

$$d(h\widehat{dt_i}) = \partial h \partial t_i dt_1 \wedge \dots \wedge dt_n.$$

Thus, J is made up of all power series $\sum a_{\mathbf{i}} t^{\mathbf{i}}$ where $p \not| (\mathbf{i}_j + 1)$ for some $1 \leq j \leq n$. In other words, $k[[x_1, \dots, x_n]]/J$ is the set of power-series of the form $\sum a_{\mathbf{j}} t^{\mathbf{p-1+pj}}$. But this is obviously identified with $(k[[t_1, \dots, t_n]])^p$, and unraveling our identifications yields the desired formula. ∎

Following Brion and Kumar, we also obtain the following:

Proposition 2.2. [BK05, Proposition 1.3.7] *Let X be a nonsingular variety. Then the following map η is an isomorphism. The map η*

$$\eta : \mathscr{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \to \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

is defined as follows: Working locally, fix a local generator ω for $\omega_{X,x}$. Furthermore, for $\psi \in \mathscr{H}om_{\mathcal{O}_{X,x}}(\omega_{X,x}, F_*\omega_{X,x})$ and $f \in \mathcal{O}_{X,x}$, we define $\eta(\psi)f$ to be the ω coefficient of $T(f\psi(\omega))$.

This is well defined and furthermore, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathscr{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) & \xrightarrow{\eta} & \mathscr{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \\ \downarrow T & & \downarrow \text{eval at 1} \\ \mathscr{H}om_{\mathcal{O}_X}(\omega_X, \omega_X) & \xrightarrow{\kappa} & \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X) \end{array}$$

where κ is the natural isomorphism.

Proof. Fix $g \in \mathcal{O}_{X,x}$. Then notice that $\eta(\psi \cdot g)$ is defined by the rule

$$T(f\psi(g\omega))/\omega = T(fg^p\psi(\omega))/\omega = gT(f\psi(\omega))/\omega$$

In particular, η is $F_*\mathcal{O}_X$-linear.

We now show that our local definition of η is well defined. Suppose that $\omega' = u\omega$ for some unit $u \in \mathcal{O}_{X,x}$. With this, we define a new map η', where $\eta'(\psi)(f) = T(f\psi(\omega'))/\omega'$. So,

$$\eta'(\psi)(f) = T(f\psi(\omega'))/\omega' = T(f\psi(u\omega))/\omega' = T(fu^p\psi(\omega))/\omega' = uT(f\psi(\omega))/(u\omega) = \eta(\psi)(f).$$

Now we show that the diagram commutes. Given $\psi \in \mathscr{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)$, the left-vertical arrow is defined by:

$$(T(\psi))(f\omega) = T(\psi(f\omega)) = fT(\psi(\omega)).$$

In particular, $\kappa(T(\psi))$ is the map obtained by multiplication by $T(\psi(\omega))/\omega$. On the other hand, the composition of η with the right vertical arrow is just

$$\eta(\psi)(1) = T(\psi(\omega))/\omega.$$

Therefore, the diagram commutes as desired.

Finally, we show that η is an isomorphism. We work locally and fix a minimal set of generators $x_1, \dots, x_n$ for the maximal ideal of $\mathcal{O}_{X,x}$. Notice that $\psi \in \mathscr{H}om_{\mathcal{O}_{X,x}}(\omega_{X,x}, F_*\omega_{X,x})$, defined by the rule

$$\psi(f dt_1 \wedge \dots \wedge dt_n) = f^p dt_1 \wedge \dots \wedge dt_n$$

This map clearly generates $\mathscr{H}om_{\mathcal{O}_X}(\omega_{X,x}, F_*\omega_{X,x})$ as an $F_*\mathcal{O}_{X,x}$-module. Now, $\eta(\psi)(f) = T(f\psi(\omega))/\omega = T(f\omega) = S(f)$. In particular, since S generates $\mathrm{Hom}_{\mathcal{O}_{X,x}}(F_*\mathcal{O}_{X,x}, \mathcal{O}_{X,x})$, we see that η is surjective, and thus it is an isomorphism since both modules are rank-1 $F_*\mathcal{O}_X$-modules. □

References

[BK05] M. Brion and S. Kumar: *Frobenius splitting methods in geometry and representation theory*, Progress in Mathematics, vol. 231, Birkhäuser Boston Inc., Boston, MA, 2005. MR2107324 (2005k:14104)

[Har77] R. Hartshorne: *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. MR0463157 (57 #3116)

[HH89] M. Hochster and C. Huneke: *Tight closure and strong F-regularity*, Mém. Soc. Math. France (N.S.) (1989), no. 38, 119–133, Colloque en l'honneur de Pierre Samuel (Orsay, 1987). MR1044348 (91i:13025)

[Laz04] R. Lazarsfeld: *Positivity in algebraic geometry. I*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 48, Springer-Verlag, Berlin, 2004, Classical setting: line bundles and linear series. MR2095471 (2005k:14001a)

[Sch09] K. Schwede: *F-adjunction*, Algebra Number Theory **3** (2009), no. 8, 907–950.

[Smi00] K. E. Smith: *Globally F-regular varieties: applications to vanishing theorems for quotients of Fano varieties*, Michigan Math. J. **48** (2000), 553–572, Dedicated to William Fulton on the occasion of his 60th birthday. MR1786505 (2001k:13007)

[Tak04] S. Takagi: *An interpretation of multiplier ideals via tight closure*, J. Algebraic Geom. **13** (2004), no. 2, 393–415. MR2047704 (2005c:13002)