F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 11/23-2010

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1. Hara's surjectivity lemma continued

Now consider the following setup:

Let D be a $\mathbb{Q}$-divisor such that $\mathrm{Supp}(\{D\}) \subseteq \mathrm{Supp}(E)$. Set $B = -p\lfloor -D \rfloor + \lfloor -pD \rfloor = p\lceil D \rceil - \lceil pD \rceil$ and note it is an effective divisor supported in E whose coefficients are between 0 and $p-1$. Therefore, $(p-1)E - B$ is also such a divisor. Thus we have a quasi-isomorphism:

$$F_*\Omega_X^{\bullet}(\log E) \subseteq F_*(\Omega_X^{\bullet}(\log E)((p-1)E - B)).$$

Therefore, composition with C^{-1} gives us an isomorphism

$$\Omega_X^i(\log E) \cong \mathcal{H}^i\left(F_*(\Omega_X^{\bullet}(\log E)((p-1)E - B))\right).$$

Twisting by $\mathcal{O}_X(-E + \lceil D \rceil)$, we get an isomorphism

$$\begin{aligned} &\Omega_X^i(\log E)(-E + \lceil D \rceil) \\ \cong\ &\mathcal{H}^i\left(F_*(\Omega_X^{\bullet}(\log E)((p-1)E - B - pE + p\lceil D \rceil))\right) \\ \cong\ &\mathcal{H}^i\left(F_*(\Omega_X^{\bullet}(\log E)(-E + \lceil pD \rceil)\right). \end{aligned}$$

We denote the ith cocycle and coboundary of $F_*(\Omega_X^{\bullet}(\log E)(-E + \lceil pD \rceil)$ by $\mathcal{Z}^i$ and $\mathcal{B}^i$ respectively. Thus we have the following sequences for all i.

$$0 \to \mathcal{Z}^i \to F_*(\Omega_X^i(\log E)(-E + \lceil pD \rceil)) \to \mathcal{B}^{i+1} \to 0$$

$$0 \to \mathcal{B}^i \to \mathcal{Z}^i \to \Omega_X^i(\log E)(-E + \lceil D \rceil) \to 0$$

The second sequence, for $i = d$, is simply

$$0 \to \mathcal{B}^d \to \mathcal{Z}^d = F_*(\Omega_X^d(\log E)(-E + \lceil pD \rceil)) = F_*\omega_X(\lceil pD \rceil) \to \omega_X(\lceil D \rceil) \to 0. \tag{1}$$

Now assume

(a) $H^j(X, \Omega_X^i(logE)(-E + \lceil D \rceil)) = 0$ for $i + j = d + 1$ and $j > 1$.
(b) $H^j(X, \Omega_X^i(logE)(-E + \lceil pD \rceil)) = 0$ for $i + j = d$ and $j > 0$.

We will prove that

$$H^0(X, F_*\omega_X(\lceil pD \rceil)) = \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X(\lfloor -pD \rfloor), \omega_X) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(\lfloor -D \rfloor), \omega_X) = H^0(X, \omega_X(\lceil D \rceil))$$

surjects.

Proof. Therefore, to show that we have our desired surjectivity, it is sufficient to show that $H^1(X, \mathcal{B}^d) = 0$. Thus, by the first short exact sequence, to show this, it is sufficient to show that $H^2(X, \mathcal{Z}^{d-1}) = 0$ and $H^1(X, F_*(\Omega_X^{d-1}(\log E)(-E + \lceil pD \rceil))) = 0$. The second of these is zero by hypothesis.

To show that $H^2(X, \mathcal{Z}^{d-1}) = 0$, by the second short exact sequence, it is sufficient to show that $0 = H^2(X, \mathcal{B}^{d-1}) = H^2(X, \Omega_X^{d-1}(\log E)(-E + \lceil D \rceil))$. The second of these is zero by hypothesis. Continuing in this way, to show that $H^2(X, \mathcal{B}^{d-1}) = 0$, it is sufficient to show that $H^3(X, \mathcal{Z}^{d-2}) = 0$, for which it is sufficient to show that $H^3(X, \mathcal{B}^{d-2}) = 0$, which eventually vanishes at $H^{d+1}(X, \mathcal{Z}^0) = 0$. □

Now, all we have to show is that our desired vanishings (a), (b) actually hold (for $p \gg 0$). For D ample (b) should hold by Serre-vanishing for p large and (a) should hold by Kodaira-Akizuki-Nakano:

Theorem 1.1. [DI87], [Har98] *Suppose that X is d-dimensional and projective over a Noetherian affine scheme, and let D be an ample $\mathbb{Q}$-divisor with $\operatorname{Supp}(\{D\}) \subseteq \operatorname{Supp}(E)$ (where E is as before, a SNC divisor). Assume that $E \subseteq X$ admits a lifting to $W_2(k)$.*[1] *Then if $i + j > d$ and $p > d$, then*

$$H^j(X, \Omega^i_X(\log E)(-E + \lceil D \rceil)) = 0.$$

Proof. The result will be a corollary of the following result of Deligne-Illusie, with notation as above we have a quasi-isomorphism of $\mathcal{O}_X$-modules:

$$\bigoplus_{i=0}^{d} \Omega^i_X(\log E)[-i] \cong F_* \Omega^\bullet_X(\log E).$$

To see this, notice that we already had a quasi-isomorphism

$$F_* \Omega^\bullet_X(\log E) \cong F_*(\Omega^\bullet_X(\log E))((p-1)E - B))).$$

Twisting by $\mathcal{O}_X(-E + \lceil D \rceil)$ gives us a quasi-isomorphism

$$\bigoplus_{i=0}^{d} \Omega^i_X(\log E)(-E + \lceil D \rceil)[-i] \cong F_* \Omega^\bullet_X(\log E)(-E + \lceil pD \rceil).$$

Taking (hyper-)cohomology, we get

$$\oplus_{i+j=m} H^j(X, \Omega^i_X(\log E)(-E + \lceil D \rceil)) \cong \mathbb{H}^m(X, \Omega^\bullet_X(\log E)(-E + \lceil pD \rceil)).$$

Remember, we are trying to show that the terms of the left side are zero for $i + j = m > d$. But we also have the Hodge-to-De Rham spectral sequence

$$E_1^{ji} := H^j(X, \Omega^i_X(\log E)(-E + \lceil pD \rceil) \Rightarrow \mathbb{H}^m(X, \Omega^\bullet_X(\log E)(-E + \lceil pD \rceil))$$

and so it suffices to show that the terms $H^j(X, \Omega^i_X(\log E)(-E + \lceil pD \rceil)$ vanish for $i + j > d$. Repeating this process, it suffices to show that the terms

$$H^j(X, \Omega^i_X(\log E)(-E + \lceil p^e D \rceil)$$

vanish for $i + j > d$ and $e \gg 0$. But this is obvious by Serre vanishing. □

We now do the following reduction to characteristic $p \gg 0$ statement.

Lemma 1.2. [Har98] *Begin with X, E, D as before, but in characteristic zero. The following vanishings hold for reduction to characteristic $p \gg 0$.*

(a) $H^j(X_p, \Omega^i_{X_p}(\log E_p)(-E_p + \lceil p^e D_p \rceil)) = 0$ *for $i + j > d$ and $e \geq 0$.*
(b) $H^j(X_p, \Omega^i_{X_p}(\log E_p)(-E_p + \lceil p^{e+1} D_p \rceil) = 0$ *for $j > 0$ and $e \geq 0$.*

[1] This means there exists a smooth scheme $\widetilde{X}$ and a SNC divisor $\widetilde{E} = \sum_i \widetilde{E}_i$ over $\operatorname{Spec} W_2(k)$ with $\widetilde{X} = X \times_k W_2(k)$ and $\widetilde{E}_i = E_i \times_k W_2(k)$.

Proof. The reason that these do not follow from standard reduction to characteristic p is because the twisting p involved depends on the actual sheaf in question. We need uniform vanishing results! Suppose A is the finitely generated $\mathbb{Z}$-algebra over which we do the reduction mod p (ie, $X_A \otimes_A \mathbb{C} = X$ and $X_A \otimes_A A/\mathfrak{p} = X_p$ for some maximal ideal $\mathfrak{p} \in \operatorname{Spec} A$).

Consider the quasi-coherent sheaf

$$\mathscr{F}_A = \bigoplus_{n \geq 0} \Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil).$$

For each j, $H^j(X_A, \mathscr{F}_A)$ is a finitely generated module of $\mathcal{R}(X_A, D_A) := \oplus H^0(X_A, \mathcal{O}_{X_A}(\lfloor nD_A \rfloor))$ which itself is a finitely generated A-algebra (remember, D_A is ample). So by generic freeness, we may assume that $\mathscr{F}_A$ is (locally) A-free, and thus each graded piece $\Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil)$ is also (locally) A-free.

Therefore,

$$H^j(X_A, \Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil)) \otimes_A A/bp = H^j(X_p, \Omega^i_{X_p}(\log E_p)(-E_p + \lceil nD_p \rceil)).$$

In particular, if the given vanishing (for a fixed n) holds for some $\mathfrak{p}$, they hold for all maximal $\mathfrak{p} \in \operatorname{Spec} A$. To prove (a), we'd need to show that the required lifting properties are satisfied, for some $\mathfrak{p}$. But for a sufficiently general $\mathfrak{p}$, the lifting properties required are satisfied!

For condition (b), we know that there exists an $n_0 \geq 0$ such that $H^j(X_A, \Omega^i_{X_A/A}(\log E_A)(-E_A + \lceil nD_A \rceil)) = 0$ for some $j > 0$ and all $n \geq n_0$. But then since the characteristic of $A/\mathfrak{p} \geq n_0$ for a Zariski-dense set of $\mathfrak{p} \in \operatorname{Spec} A$, we are done. □

References

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