F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 11/18-2010

KARL SCHWEDE

1. Finitistic test ideals, tight closure for modules, and tight closure of pairs

Let us prove another variant of this below, first however, a lemma.

Lemma 1.1. *Suppose that R is a d-dimensional F-finite local domain. Then $H^d_{\mathfrak{m}}(R) \otimes F^e_* R$ is naturally identified with $H^d_{\mathfrak{m}}(F^e_* R)$.*

Proof. Choose a system of parameters $x_1, \ldots, x_d$ for R, and compute local cohomology in terms of the Čech complex with respect to those parameters. $H^d_{\mathfrak{m}}(R)$ is then identified with the cokernel of the map

$$\oplus R_{\hat{x_i}} \to R_{x_1 \ldots x_d}.$$

Tensoring that map with $F^e_* R$, gives us the term of the Čech complex corresponding to the system of parameters $x_1^{p^e}, \ldots, x_d^{p^e}$. This completes the proof, in fact one also sees that $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R) \otimes F^e_* R$ is identified with $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(F^e_* R)$. □

Proposition 1.2. [Smi97] *Suppose that R is a d-dimensional F-finite local domain. Then the tight closure of zero in $H^d_{\mathfrak{m}}(R)$ is the unique largest non-zero module $M \subseteq H^d_{\mathfrak{m}}(R)$ such that $F(M) \subseteq M$ where $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R) = F_* H^d_{\mathfrak{m}}(R) = H^d_{\mathfrak{m}}(F_* R)$ is the map induced by Frobenius.*

Proof. For simplicity, we assume that R is complete, in the general case use the faithfull flatness of $\mathrm{Hom}_R(\_\_, E)$. First we show that $F(0^*_{H^d_{\mathfrak{m}}(R)}) \subseteq 0^*_{H^d_{\mathfrak{m}}(R)}$. Suppose that $z \in 0^*_{H^d_{\mathfrak{m}}(R)}$. Thus there exists $c \in R$ such that $0 = cz^{p^e} \in H^d_{\mathfrak{m}}(R) \otimes F^e_* R$ for all $e \geq 0$ (by the previous lemma, we need not be careful about tensor products). Then $0 = c^p (z^p)^{p^e} \in H^d_{\mathfrak{m}}(R)$, so $F(z) \in 0^*_{H^d_{\mathfrak{m}}(R)}$.

Now suppose that N is any proper submodule of $H^d_{\mathfrak{m}}(R)$ such that $F(N) \subseteq N$. We know that $T := \mathrm{Hom}_R(H^d_{\mathfrak{m}}(R)/N, E) \subseteq \mathrm{Hom}_R(H^d_{\mathfrak{m}}(R), E) = \omega_R$. But ω_R is rank-one, so there exists a $c \in R$ such that $c\omega_R \subseteq T$, thus we have the composition

$$c\omega_R \subseteq T \subseteq \omega_R.$$

Dualizing again, we get

$$H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)/N \to cH^d_{\mathfrak{m}}(R)$$

where the composition is multiplication by c. This implies that N is annihilated by c. Thus if $z \in N$, $cz^{p^e} = cF^e(z) \in cF^e(N) \subseteq cN = 0$ for all $e \geq 0$, implying that $z \in 0^*_{H^d_{\mathfrak{m}}(R)}$ and completing the proof. □

Finally, we briefly define tight closure of pairs.

Definition 1.3. [Tak04], [HY03], [Sch08b], [Sch08a], [HH90] Suppose R is an F-finite domain, $X = \operatorname{Spec} R$ and $(X, \Delta, \mathfrak{a}^t)$ is a triple. Further suppose that M is a (possibly non-finitely generated) R-module and that N is a submodule of M. We say that an element $z \in M$ is in the $(\Delta, \mathfrak{a}^t)$-*tight closure of* N *in* M, denoted $N_M^{*\Delta,\mathfrak{a}^t}$, if there exists an element $0 \neq c \in R$ such that, for all $e \gg 0$ and all $a \in \mathfrak{a}^{\lceil t(p^e-1)\rceil}$, the image of z via the map

$$(F^e_* i) \circ \mathbb{F}^e_*(\times ca) \circ F^e : M \longrightarrow M \otimes_R F^e_* R \xrightarrow{F^e_*(\times ca)} M \otimes_R F^e_* R \longrightarrow M \otimes_R F^e_* R(\lceil (p^e-1)\Delta \rceil)$$

is contained in $N_M^{[q]\Delta}$, where we define $N_M^{[q]\Delta}$ to be the image of $N \otimes_R F^e_* R(\lceil (p^e-1)\Delta\rceil)$ inside $M \otimes_R F^e_* R(\lceil (p^e-1)\Delta\rceil)$.

Most of the theory of test elements / ideals can be generalized to this setting, although some of the arguments used so far do not work. See [HY03], [Tak04], [Sch08b] and [Sch08a] for some additional discussion.

2. Hara's surjectivity lemma

Our goal is to show the following theorem.

Lemma 2.1. [Har98] *Suppose that* R_0 *is a ring of characteristic zero,* $\pi : \widetilde{X}_0 \to \operatorname{Spec} R_0$ *is a log resolution of singularities,* D_0 *is a* π*-ample* $\mathbb{Q}$*-divisor with simple normal crossings support. We reduce this setup to characteristic* $p \gg 0$*. Then the natural map*

$$(F^e)^\vee = \Phi_{\widetilde{X}} : F^e_* \omega_{\widetilde{X}}(\lceil p^e D\rceil) \to \omega_{\widetilde{X}_p}(\lceil D \rceil)$$

surjects.

We will show it in the following way. We follow Hara's proof.

Proposition 2.2. *Suppose that* X *is a* d*-dimensional smooth variety (quasi-projective) of finite type over a perfect field* k *of characteristic* $p > 0$*.* [1] *Further suppose that* $E = \sum E_j$ *is a reduced simple normal crossings divisor on* X*. Suppose in addition that* D *is a* $\mathbb{Q}$*-divisor on* X *such that* $\operatorname{Supp}(D - \lfloor D \rfloor) = \operatorname{Supp}(\{D\}) \subseteq \operatorname{Supp}(E)$*.*

Additionally, suppose that the following two vanishings hold:

(a) $H^j(X, \Omega^i_X(logE)(-E - \lfloor -D \rfloor)) = 0$ *for* $i + j = d + 1$ *and* $j > 1$*.*
(b) $H^j(X, \Omega^i_X(logE)(-E - \lfloor -pD \rfloor)) = 0$ *for* $i + j = d$ *and* $j > 0$*.*

Then, the natural map

$$H^0(X, F_*\omega_X(\lceil pD\rceil)) = \operatorname{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X(\lfloor -pD\rfloor), \omega_X) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(\lfloor -D \rfloor), \omega_X) = H^0(X, \omega_X(\lceil D \rceil))$$

surjects.

Our plan is as follows:

(i) Prove the proposition.
(ii) Show for an ample $\mathbb{Q}$-divisor D reduced from characteristic $p \gg 0$, conditions (a) and (b) hold.
(iii) The e-iterated version of Hara's lemma will then follow from composing the surjectivity from the proposition and composition of maps.

[1] We may as well assume $k = \mathbb{F}_p$ for simplicity, we'll only want this for finite fields, and all the arguments are essentially the same as over $\mathbb{F}_p$.

In order to prove the proposition, we will need to briefly recall the Cartier operator. From here on out, X and E are as in Proposition 2.2. Consider the (log)de-Rham complex, $\Omega^{\bullet}_X(\log E)$. This is not a complex of $\mathcal{O}_X$-modules (the differentials are not $\mathcal{O}_X$-linear). However, the complex

$$F_*\Omega^{\bullet}_X(\log E)$$

is a complex of $\mathcal{O}_X$-modules (notice that $d(x^p) = 0$).

Definition-Proposition 2.3. [Car57], [Kat70] [cf [EV92], [BK05]] There is a natural isomorphism (of $\mathcal{O}_X$-modules):

$$C^{-1} : \Omega^i_X(\log E) \to \mathcal{H}^i(F_*\Omega^{\bullet}_X(\log E))$$

Furthermore, $(C^{-1})^{-1}$ for $i = d$ and $E = 0$, induces a map $F_*\omega_X \to \mathcal{H}^d(F_*\Omega^{\bullet}_X(\log E)) \cong \omega_X$ which corresponds to the natural dual of Frobenius[2].

Let us explain how to construct this isomorphism C^{-1}. We follow [EV92, 9.13] and [Kat70]. We begin with C^{-1} in the case that $i = 1$ and $E = 0$. We work locally on X (which we assume is affine) and we define C^{-1} by its action on $dx \in \Omega^i_X(\log E)$, $x \in \mathcal{O}_X$; $C^{-1}(dx) = x^{p-1}dx$ (or rather, its image in cohomology). In the $E \neq 0$ case, if t is a local parameter of E, then we define $C^{-1}(\frac{dt}{t}) = dt/t$.

We should show that C^{-1} is additive, we start in the $E = 0$ case. First notice that $d(x^{p-1}dx) = 0$ so at least the image of $x^{p-1}dx$ is in the cohomology of the de Rham complex.

Now, $C^{-1}(d(x) + d(y)) = C^{-1}(d(x + y)) = (x + y)^{p-1}d(x + y)$, we need to compare this to $x^{p-1}dx + y^{p-1}dy$. Write $f = \frac{1}{p}\left((x + y)^p - x^p - y^p\right)$ (where the $\frac{1}{p}$ just formally cancels out the ps in the binomial coefficients). Then

$$df = d\sum_{i,j>0,i+j=p} \gamma_i x^i y^{p-i} = \left(\sum_{i>0,j>0,i+j=p-1} \gamma_i i x^{i-1}y^{p-i}\right) dx + \left(\sum_{i>0,j>0,i+j=p-1} \gamma_i p - i x^i y^{p-i-1}\right) dy$$

where $\gamma_i = \frac{1}{p}\binom{p}{i} = \frac{(p-1)(p-2)\dots 1}{i!(p-i)!} = \frac{1}{p-i}\binom{p-1}{i} = \frac{1}{i}\binom{p-1}{p-i}$. Thus

$$df = (x + y)^{p-1}(dx + dy) - x^{p-1}dx - y^{p-1}dy.$$

Therefore, $x^{p-1}dx + y^{p-1}dy$ and $(x + y)^{p-1}d(x + y)$ are the same in cohomology.

For the $E \neq 0$ case and t a defining equation of a component of E, simply observe that

$$C^{-1}(dt) = C^{-1}\left(t\frac{dt}{t}\right) = t^p C^{-1}\left(\frac{dt}{t}\right) = t^p\frac{dt}{t} = t^{p-1}dt,$$

which at least shows that the definition of C^{-1} we gave is compatible, the additivity follows.

We define C^{-1} for $i > 1$ using wedge powers of C^{-1} for $i = 1$. We should also show that all these C^{-1} are isomorphisms. For simplicity, we work with the case that $X = \mathbb{F}_p[x, y]$ and $E = 0$ (see [EV92] or [Kat70] for how to reduce the polynomial ring case in general), let us explicitly see that the first C^{-1} is an isomorphism.

First we show that C^{-1} is injective. Suppose that $C^{-1}(fdx + gdy) = 0$, which means $C^{-1}(fdx + gdy) = dh$ for some $h \in \mathcal{O}_X$. Thus $f^p x^{p-1}dx + g^p y^{p-1}dy = dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy$. Now, we know $f^p x^{p-1} = \sum \lambda_{i,j} y^{ip} x^{jp+p-1} = \frac{\partial h}{\partial x}$, but this is ridiculous because we claim that

[2] This is important, it gives us a "canonical" map between these two modules (before it was always defined up to multiplication by units)

this is the derivative of some h with respect to x. If you take a derivative of some polynomial in x with respect to x, no output can ever have x^{jp+p-1} in it.

The surjectivity of C^{-1} is more involved. See for example, [], [] or [], and follows similar lines to the proof of the next lemma. The isomorphism of the higher C^{-1} is an application of the Künneth formula.

We also need the following lemma.

Lemma 2.4. [Har98, Lemma 3.3] *With notation as in Proposition 2.2, additionally let $B = \sum r_j E_j$ be an effective integral divisor supported on E such that each $0 \leq r_j \leq p-1$. It follows that the inclusion of complexes (of $\mathcal{O}_X^p$-modules)*

$$\Omega_X^{\bullet}(\log E) \hookrightarrow (\Omega_X^{\bullet}(\log E))(B) := (\Omega_X^{\bullet}(\log E)) \otimes_{\mathcal{O}_X} \mathcal{O}_X(B)$$

is a quasi-isomorphism.

Proof. First we explain the differential on $(\Omega_X^{\bullet}(\log E))(B)$ because the tensor product with B is as an $\mathcal{O}_X$-module, it is not so clear what the differential is. However, we simply restrict the differential from $i_* \Omega_{X \setminus E}^{\bullet}$ to $(\Omega_X^{\bullet}(\log E))(B)$.

Now, the question is local, so we assume that X is the spectrum of a local ring. Choose $t_1, \ldots, t_d$ to be local parameters (which also form a p-basis), where the components E_i of E are defined by $t_1, \ldots, t_r$ respectively. Consider the complexes:

$$\mathscr{K}_j^{\bullet} = \left[0 \to \bigoplus_{i=0}^{p-1} t_j^i \mathcal{O}_X^p \to \bigoplus_{i=0}^{p-1} (t_j^i \frac{dt_j}{t_j^{\varepsilon_j}}) \mathcal{O}_X^p \right]$$

where the middle-map is the usual d and where $\varepsilon_j = 1$ if $j \leq r$ and is zero otherwise. Set

$$\mathscr{J}_j^{\bullet} = t_j^{-r_j} \mathscr{K}_j^{\bullet},$$

for $j \leq r$.

We certainly have inclusions $\mathscr{K}_j^{\bullet} \subseteq \mathscr{J}_j^{\bullet}$, we claim that these are actually quasi-isomorphisms. We work in a very specific case, that of $k[x,y]$ where $E = \div X$. We only look at $\mathscr{K}_1$, of course the general case is exactly the same. We have the inclusion of complexes:

$$\begin{array}{ccc} \oplus_{i=0}^{p-1} x^i \mathcal{O}_X^p & \longrightarrow & \bigoplus_{i=0}^{p-1} x^i \frac{dx}{x} \\ \big\downarrow & & \big\downarrow \\ \oplus_{i=0}^{p-1} x^{i-r} \mathcal{O}_X^p & \longrightarrow & \bigoplus_{i=0}^{p-1} x^{i-r-1} dx. \end{array}$$

One can easily verify that the cokernel and kernel of the two rows "line-up" because r is between 0 and $p-1$. Thus we have proved our claim.

Now, we claim that

$$\Omega_X^{\bullet}(\log E) = \mathscr{K}_1^{\bullet} \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^{\bullet} \otimes \ldots \otimes_{\mathcal{O}_X^p} \mathscr{K}_d^{\bullet}.$$

We'll check this for $X = \operatorname{Spec} \mathbb{F}_p[x, y]$ and $E = 0$. Here $\mathscr{K}_1 = \left[\bigoplus_{i=0}^{p-1} x^i \mathcal{O}_X^p \to \bigoplus_{i=0}^{p-1} (x^i dx) \mathcal{O}_X^p\right]$, and likewise $\mathscr{K}_2 = \left[\bigoplus_{i=0}^{p-1} y^i \mathcal{O}_X^p \to \bigoplus_{i=0}^{p-1} (y^i dy) \mathcal{O}_X^p\right]$. Thus $\mathscr{K}_1^\bullet \otimes \mathscr{K}_2^\bullet$ is the complex associated to the double-complex

$$\mathscr{K}_1^1 \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^0 \cong (dx)\mathcal{O}_X \qquad \mathscr{K}^1 \otimes_{\mathcal{O}_X^p} \mathscr{K}^2 \cong (dx \wedge dy)\mathcal{O}_X$$

$$\mathscr{K}_1^0 \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^0 \cong \mathcal{O}_X ar[u] \longrightarrow \mathscr{K}_1^0 \otimes_{\mathcal{O}_X^p} \mathscr{K}_2^1 \cong (dy)\mathcal{O}_X$$

The general case is similar, but messy to write down.

Arguing similarly, we have that

$$\Omega_X^\bullet(\log E)(B) \cong \mathscr{J}_1^\bullet \otimes \dots \mathscr{J}_r^\bullet \otimes \mathscr{K}_{r+1}^\bullet \otimes \dots \mathscr{K}_d^\bullet$$

and we have the natural (compatible) inclusion $\Omega_X^\bullet(\log E) \to \Omega_X^\bullet(\log E)(B)$ which are quasi-isomorphisms by the Künneth formula. □

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