F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 11/16-2010

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1. Tight closure

Definition 1.1. A *finitistic test element* $0 \neq c \in R$, is an element of R such that for every ideal I and every $z \in I^*$,

$$cz^{p^e} \in I^{[p^e]}$$

for all $e \geq 0$.

It should be highly unclear that such a test element exists. However, we have already shown the following lemma.

Lemma 1.2. *Given an F-finite domain R, there exists $0 \neq c \in R$ such that for every $0 \neq d \in R$, $c \in \phi(dR)$ for some $\phi : F^e_* R \to R$.*

Corollary 1.3. *The c in the above lemma is a finitistic test element.*

Proof. Suppose that $0 \neq d \in R$ is an element of R such that $dz^{p^e} \in I^{[p^e]}$ for all $e > 0$, it follows from the statement above that there exists $\phi : F^a_* R \to R$ such that $\phi(d) = c$. Thus, for $e \geq a$,

$$cz^{p^e} = \phi(dz^{p^{e+a}}) \in \phi\left(I^{[p^{e+a}]}\right) \subseteq I^{[p^e]}.$$

$\square$

Definition 1.4. The *finitistic test ideal* $\tau_f(R)$ is defined to be the ideal of R generated by all finitistic test elements. It can also be described as the set made up of all finitistic test elements and zero.

Lemma 1.5. *We have $\tau_f(R) = \cap_{I \subseteq R}(I : I^*)$.*

Proof. Suppose that $c \in \tau_f R$, then $cz^{p^e} \in I^{[p^e]}$ for all $e \geq 0$, in particular for $e = 0$. Thus $cz \in I$ and $c \in \cap_{I \subseteq R}(I : I^*)$.

Conversely, suppose that $c \in \cap_{I \subseteq R}(I : I^*)$. Choose $z \in I^*$. Then I claim that $z^{p^a} \in (I^{[p^a]})^*$ for all $a \geq 0$. But $cz^{p^e} \in I^{[p^e]}$ for all $e \geq 0$ so that $c^{p^a}(z^{p^a})^{p^e} \in (I^{[p^a]})^{[p^e]}$ for all a, and the claim is proven. Thus $cz^{p^a} \in I^{[p^a]}$ for all $a \geq 0$ because c was chosen in the intersection, which implies that c is a finitistic test element. $\square$

Corollary 1.6. *R is weakly F-regular if and only if $\tau_f(R) = R$.*

We now come to the proof of Briançon-Skoda theorem via tight closure.

Theorem 1.7. [] *Let R be an F-finite domain, and $(u_1, \ldots, u_n) = I \subseteq R$ an ideal. Then for every natural number m,*

$$\overline{I^{m+n}} \subseteq \overline{I^{m+n-1}} \subseteq (I^m)^*$$

and so

$$\tau(R)\overline{I^{m+n}} \subseteq I^m.$$

which gives a very nice statement in the case that R is F-regular (and so $\tau(R) = R$).

This proof is taken from []. For any $y \in \overline{I^{m+n-1}}$, we know that there exists $0 \neq c \in R$ such that $cy^l \in (I^{m+n-1})^l$ for all $l \geq 0$. Consider a monomial $u_1^{a_1} \dots u_n^{a_n}$ where $a_1 + \cdots + a_n = l(m+n-1)l$. Write each $a_i = b_i l + r_i$ where $0 \leq r_i \leq l-1$. We claim that the sum of the b_i is at least m, which will imply that the monomial is contained in $(I^m)^{[l]}$ for all l such that $l = p^e$. However, if the sum $b_1 + \cdots + b_m \leq m-1$, then $l(m+n-1) = \sum a_i \leq l(m-1) + n(l-1) = l(m+n-1) - n < l(m+n-1)$, which implies the claim.

Thus $cy^{p^e} \in (I^m)^*$ as desired. □

Remark 1.8. Previously, in the proof that test ideals and multiplier ideals coincided after reduction mod $p \gg 0$, we used this theorem on $\overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil + r}}$ where r is the number of generators of $\mathfrak{a}$. The tight-closure Briançon-Skoda theorem tells us that this is contained in $\mathfrak{a}^{\lceil t(p^e-1)\rceil}$.

1.1. Hilbert-Kunz(-Monsky) multiplicity.

Recall the following definition:

Definition 1.9. Suppose that $(R, \mathfrak{m})$ is a d-dimensional local ring and I is an $\mathfrak{m}$-primary ideal. We define the *multiplicity of R (at I)* to be

$$e(I, R) := \lim_{n \to \infty} \frac{d!(R/I^n)}{n^d}.$$

Note that R is regular if and only if $e(\mathfrak{m}, R) = 1$.

Using this as a guide, Kunz considered the following notion.

Definition 1.10. [Kun69], [Mon83] Suppose that $(R, \mathfrak{m})$ is a d-dimensional local ring. We define the *Hilbert-Kunz-Monsky multiplicity of R (at $\mathfrak{m}$)* to be

$$e_{HKM}(I, R) := \lim_{n \to infty} \frac{(R/I^{[p^e]})}{p^{ed}}$$

Kunz showed that $e_{HKM}(\mathfrak{m}, R) = 1$ if R is regular (we basically also did in the first few days of class), and Watanabe-Yoshida [WY00] (and Huneke-Yao, [HY02]) showed the converse.

Remark 1.11. In fact, this $e(I, R)$ can be viewed as some sort of leading coefficient of a polynomial computing (R/I^n). While it is true that $(R/I^{[p^e]}) = e_{HKM}(I, R)p^{ed} + O(p^{e(d-1)})$, the lower order terms are not generally a polynomial, unlike $e(I, R)$

Kunz actually thought that this limit didn't exist, and even had a claimed counter-example. (Un?)Fortunately, there was a mistake and Monsky later showed that the limit did indeed exist. The reason we mention it now is the following theorem of Hochster-Huneke.

Theorem 1.12. [HH90] *Suppose $(R, \mathfrak{m})$ is an equidimensional F-finite local domain. Further suppose that $I \subseteq J$ are two $\mathfrak{m}$-primary ideals. Then if $J \subseteq I^*$ if and only if $e_{HKM}(I, R) = e_{HKM}(J, R)$.*

Proof. We will only prove one direction, for the converse, see [HH90]. Suppose then that $J \subseteq I^*$, in other words, suppose that $I^* = J^*$. We first show that there exists a $c \in R^\circ$ such that $cJ^{[q]} \subseteq I^{[q]}$ for all $q \gg 0$. But this is easy, choose a set of generators $x_1, \dots, x_k$ of J. Then by hypothesis, there exists a $c_i \in R$ such that $c_i x_i^q \in I^{[q]}$ for all $q \gg 0$. Let c be the product of the c_i and note that $cx_i^q \in I^{[q]}$ for all $q \gg 0$. Therefore, $J^{[q]}/I^{[q]}$ is a module with at most k generators over $R/(I^{[q]} + (c))$. Set $S = R/(c)$. Thus $J^{[q]}/I^{[q]}$ is a module with at most k generators over $S/(IS)^{[q]}$. Note that $\dim S \leq \dim R - 1$.

But now we know that there is a constant C_S such that $\lambda(S/(IS)^{[q]} \leq C_S q^{d-1}$ (since Hilbert-Kunz multiplicities exist). However, we can also map $(S/(IS)^{[q]})^{\oplus k}$ onto $J^{[q]}/I^{[q]}$. Therefore,

$$\lambda(J^{[q]}/I^{[q]}) \leq kC_S q^{d-1} h^{d-1}.$$

Thus $\lambda(R/J^{[q]}) - \lambda(R/I^{[q]}) \leq Cq^{d-1}$ for $C = kC_S h^{d-1}$.

Therefore the J and I have the same Hilbert-Kunz multiplicity. $\square$

2. Finitistic test ideals, tight closure for modules, and tight closure of pairs

Definition 2.1. [HH90] Given a domain R and R-modules $N \subseteq M$, we consider the natural map

$$\gamma_e : M \to M \otimes F^e_* R$$

for each e. We say that $z \in M$ is in the *tight closure of* N *in* M if there exists a $c \in R \setminus \{0\}$ such that for all $e \geq 0$, $\gamma_e(z).c = z \otimes c$ is contained in the image of $N \otimes F^e_* R \to M \otimes F^e_* R$.

Remark 2.2. Suppose that $M = R$ and N is an ideal. Then the image of $N \otimes_R F^e_* R$ inside $R \otimes_R F^e_* R = F^e_* R$ is simply $N^{[p^e]}$. Thus this definition of tight closure coincides with the usual one.

The case we are going to be primarily concerned with is when $N = 0 \subseteq M$. Generally speaking, one can always reduce to studying this case by the following trick.

Lemma 2.3. *Suppose $N \subseteq M$ is as above, then $z \in N^*_M$ if and only if $\bar{z} \in 0^*_{M/N}$.*

Proof. Now, $z \in N^*_M$ if and only if there exists $0 \neq c \in R$ such that

$$\gamma_e(z) \otimes c \in \mathrm{Image}\left(N \otimes F^e_* R \to M \otimes F^e_* R\right).$$

But this happens if and only if $\gamma_e(\bar{z}) = 0 \subseteq (M/N) \otimes F^e_* R$ by right exactness of tensor. $\square$

Remark 2.4. In general, given $N \subseteq M \subseteq M'$, one has $N^*_M \subsetneq N^*_{M'}$. The problem is that $\otimes$ is not left-exact.

Lemma 2.5. *Suppose that R is strongly F-regular, then for every R-modules $N \subseteq M$, $N = N^*_M \subseteq M$.*

Proof. Suppose that $z \in N^*_M$. Thus there exists a $0 \neq d \in R$ such that $z \otimes d$ is contained in the image of $N \otimes F^e_* R \to M \otimes F^e_* R$ for all $e \geq 0$. Choose $\phi : F^a_* R \to R$ which sends $d \mapsto 1$. We have the following diagram

$$\begin{tikzcd}
N \otimes F^a_* R \arrow[r, "\mathrm{id}_N \otimes \phi"] \arrow[d, "f"'] & N \arrow[d, hook, "g"'] \\
M \otimes F^a_* R \arrow[r, "\mathrm{id}_M \otimes \phi"'] & M
\end{tikzcd}$$

We know that $z \otimes d$ is in the image of f, let ζ be an element of $N \otimes F^a_* R$ which maps to it. Thus

$$g\left((\mathrm{id}_N \otimes \phi)(\zeta)\right) = (\mathrm{id}_M \otimes \phi)(z \otimes d) = z$$

But g is simply the inclusion of N into M which implies that $z \in N$ as desired. $\square$

We also have the converse statement.

Proposition 2.6. [HH90], [Hoc07] *Suppose R is an F-finite local domain and that for every R-module $N \subseteq M$, $N = N^*_M$, then R is strongly F-regular.*

Proof. Let E denote the injective hull of the residue field $R/\mathfrak{m}$. We know $0^*_E = 0$ by assumption. We will show that R is strongly F-regular.

By hypothesis, $0^*_E = 0$. Choose $c \in R = F^e_* R$ and consider the map $R \to F^e_* R$ which sends $1 \mapsto c$. Tensoring with E, gives us a map $\gamma_{e,c} : E \to E \otimes_R F^e_* R$ which sends z to $z \otimes c$. Now recall that we have an isomorphism $F^e_* R \otimes \operatorname{Hom}(R, E) \cong F^e_* R \otimes_R E \cong \operatorname{Hom}_R(\operatorname{Hom}_R(F^e_* R, R), E)$ defined by the map which sends $r \otimes \phi$ to the map $h : \operatorname{Hom}_R(F^e_* R, R) \to E$ defined by the rule $h(\alpha) = \phi(\alpha(r))$. Thus $E \to E \otimes_R F^e_* R$ is identified with

$$E \cong \operatorname{Hom}_R(\operatorname{Hom}_R(R, R), E) \to \operatorname{Hom}_R(\operatorname{Hom}_R(F^e_* R, R), E).$$

The map is just induced by the inclusion $R \subseteq F^e_* R$ in the first entry which sends 1 to c. Apply $\operatorname{Hom}_R(\_\_, E)$ and Matlis duality. This gives us a map $\operatorname{Hom}_{\hat{R}}(F^e_* \hat{R}, \hat{R}) \to \operatorname{Hom}_{\hat{R}}(\hat{R}, \hat{R}) \cong \hat{R}$ induced by evaluation at c. In particular, $\gamma_{e,c}$ is injective if and only if the evaluation-at-c-map $\operatorname{Hom}_R(F^e_* R, R) \to R$ is surjective (we can remove the completion signs due to faithful flatness).

Consider now $c = 1$, we know that for any $z \in E$, $0 \neq z \otimes 1 \in E \otimes F^e_* R$ for infinitely many $e > 0$. But if it holds for infinitely many $e > 0$, then it holds for all $e \geq 0$ since $\gamma_{e,1}$ factors through $\gamma_{e-1,1}$. Therefore, $\gamma_{e,1}$ is injective and R is F-split.

Now, again consider $\gamma_{e,c}$. $\gamma_{e,c}$ is injective if and only if it is non-zero on the socle[1] Suppose that $z \in \ker(\gamma_{e,c})$, in other words $0 = z \otimes c \in E \otimes F^e_* R$. We claim that then also $z \in \ker(\gamma_{e-1,c})$. However, the composition

$$\begin{array}{ccccc} E & \xrightarrow{g} & E \otimes F^{e-1}_* R & \xrightarrow{f} & E \otimes F^e_* R \\ & & & & \\ z & \longmapsto & z \otimes c & \longmapsto & z \otimes c^p, \end{array}$$

is certainly zero, and since the map f is injective (because R is F-split), this implies that $g(z) = 0$.

Therefore, the set of kernels of $\gamma_{e,c}$ are a descending sequence of modules in E, an artinian module. Therefore they eventually stabilize. However, no element is in all the kernels because $0^*_E = 0$. Thus some evaluation-at-c-map $\operatorname{Hom}_R(F^e_* R, R) \to R$ is surjective, proving that R is strongly F-regular. $\square$

Generally speaking, using the same method as above, one can show that $\operatorname{Ann}_R 0^*_E = \tau(R)$, see for example [LS01]. In fact, any non-zero element of $\tau(R)$ can be used to "test" tight closure in any module. Furthermore, $\tau(R)$ is generated by exactly the elements $c \in R$ such that $cN^*_M \subseteq N$ for all modules $N \subseteq M$, see [Hoc07].

Conjecture 2.7. *The (big/non-finitistic) test ideal $\tau(R)$ is equal to the finitistic test ideal $\tau_f(R)$.*

[1] The 1-dimensional submodule of E which is annihilated by $\mathfrak{m}$.

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