F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 11/11-2010

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1. Vanishing theorems via finite maps and direct summand conditions

Using the methods discussed previously, one can show the following.

Proposition 1.1. [HH92] *[Also see* [Smi97] *and erratum on Smith's webpage] Suppose that X is a projective variety of characteristic $p > 0$ and that $\mathscr{L}$ is an ample line bundle. Then there exists a finite map $f : Y \to X$ such that $H^i(X, \mathscr{L}^{-j}) \to H^i(Y, f^*\mathscr{L}^{-j})$ is zero for all $0 < i < \dim X$ and all j.*

The only interesting part of this statement is the case when $j = 0$ (just take f to be a high power of the Frobenius), and the idea of the proof is the same as the equational lemma. Recently Bhargav Bhatt, see

http://www-personal.umich.edu/~bhattb/math/ddscposchar.pdf

has shown that we can extend this result in the following way

Theorem 1.2 (Bhatt)**.** *[http://www-personal.umich.edu/~bhattb/math/ddscposchar.pdf] Suppose that X is a projective variety of characteristic $p > 0$ and $\mathscr{L}$ is a semi-ample line bundle. Then there exists a finite map $f : Y \to X$ such that*

- *$H^i(X, \mathscr{L}) \to H^i(Y, f^*\mathscr{L})$ is zero for $i > 0$.*
- *If in addition, $\mathscr{L}$ is big, then we can force $H^i(X, \mathscr{L}^{-1}) \to H^i(Y, f^*\mathscr{L})$ to be zero for $i < \dim X$.*

I'll leave you to find the proofs on the web.

Bhatt was actually interested in the following. Consider the following condition on a ring R.

Definition 1.3. Suppose that R is F-finite normal domain. We say that R is a splinter (or DSCR = direct summand condition ring) if $R \subseteq S$ splits as a map of R-modules for every finite extension $R \subseteq S$. Furthermore, we say that R is a DDSCR (= derived direct summand condition ring) if $R \subseteq Rf_*\mathcal{O}_Y$ splits as a map of objects in $D^b_{\mathrm{coh}}(R)$ for every generically finite proper map $f : Y \to \operatorname{Spec} R$.

Bhatt's main result follows:

Theorem 1.4 (Bhatt)**.** *A ring in characteristic $p > 0$ is a DSCR (= splinter) if and only if it is a DDSCR.*

Again, I'll refer you to his paper for the reference.

The following is the most important conjecture in tight closure theory (or a variant of it).

Conjecture 1.5. *A ring R satisfies the DSCR if and only if it is strongly F-regular.*

This conjecture is known in the $\mathbb{Q}$-Gorenstein case, see [Sin99], [HH94].

The implication that strongly F-regular implies DSCR is easy, we prove it below.

Lemma 1.6. *Suppose that R is strongly F-regular, then R is a splinter/DSCR.*

Proof. Given a finite extension $R \subseteq S$, fix $\phi : F^e_* R \to R$. This map induces a map $\operatorname{Hom}_R(S, F^e_* R) \to \operatorname{Hom}_R(S, R)$. We also have

$$F^e_* \operatorname{Hom}_R(S, R) = \operatorname{Hom}_{F^e_* R}(F^e_* S, F^e_* R) \to \operatorname{Hom}_R(F^e_* S, F^e_* R) \to \operatorname{Hom}_R(S, F^e_* R)$$

giving us a map $F^e_* \operatorname{Hom}_R(S, R) \to \operatorname{Hom}_R(S, R)$. One can check that this induces a commutative diagram.

$$\begin{array}{ccc} F^e_* \operatorname{Hom}_R(S, R) & \longrightarrow & \operatorname{Hom}_R(S, R) \\ \downarrow & & \downarrow \\ F^e_* R & \xrightarrow[\phi]{} & R \end{array}$$

where the vertical maps are evaluation-at-1. In particular, the image of $\operatorname{Hom}_R(S, R) \to R$ is ϕ-stable, and so if R is strongly F-regular, that map is surjective. $\square$

In mixed characteristic, one can ask a related question.

Conjecture 1.7 (Hochster)**.** *Suppose that R is a regular (local) ring in mixed characteristic. Is it true that for every finite extension $R \subseteq S$, one has that the evaluation-at-1 map $\operatorname{Hom}_R(S, R) \to R$ surjects (in other words, $R \subseteq S$ splits).*

This is probably the most important conjecture in commutative algebra. This conjecture is known up through dimension 3 and is closely related to a pantheon of other conjectures known as the homological conjectures. It is obvious in dimension 1 (1-dimensional regular local rings being PIDs). Let me prove it in dimension 2.

Proposition 1.8. *Suppose that R is a regular local ring and that $R \subseteq S$ is a finite extension. Then $R \subseteq S$ splits as a map of R-modules.*

Proof. It is sufficient to prove the result in the case that S is normal and reduced and so we assume that. Choose $f \in R$ such that R/f is regular (and 1-dimensional). Consider the following diagram (we do Elkik's proof yet again).

$$\begin{array}{ccc} \omega_S = \operatorname{Hom}_R(S, R) & \longrightarrow & R = \omega_R \\ \downarrow \times f & & \times f \downarrow \\ \omega_S = \operatorname{Hom}_R(S, R) & \longrightarrow & R = \omega_R \\ \downarrow & & \downarrow \\ \omega_{S/f} & \longrightarrow & R/f = \omega_{R/f} \\ \downarrow & & \downarrow \\ 0 & & 0 \end{array}$$

where the bottom zeros exist because the rings in question are Cohen-Macaulay.

Now, $R/f \to S/f$ is a finite extension (S/f may not be reduced, but this doesn't matter), and so it splits because R/f is regular. Thus $\omega_{S/f} \to \omega_{R/f}$ surjects. In particular, our diagram becomes.

$$\begin{array}{ccccccc}
\omega_S = \operatorname{Hom}_R(S,R) & \longrightarrow & R = \omega_R & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow{\scriptstyle \times f} & & {\scriptstyle \times f}\downarrow & & \downarrow{\scriptstyle \times f} & & \\
\omega_S = \operatorname{Hom}_R(S,R) & \longrightarrow & R = \omega_R & \longrightarrow & C & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\omega_{S/f} & \longrightarrow & R/f = \omega_{R/f} & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}$$

Nakayama's lemma again implies that C is zero. ☐

Remark 1.9. In fact, one can show that for a regular ring in mixed characteristic for any generically finite map $f_*Y \to \operatorname{Spec} R$, $f_*\omega_Y \to \omega_R$ surjects.

2. Tight closure

Suppose that $R \subseteq S$ is an extension of rings. Consider an ideal $I \subseteq R$ and its extension IS. We always have that $(IS) \cap R \supseteq I$, however:

Lemma 2.1. *With $R \subseteq S$ as above and further suppose the extension splits as a map of R-modules. Then*

$$(IS) \cap R = I.$$

Proof. Fix $\phi : S \to R$ to be the splitting given by hypothesis. Suppose that $z \in (IS) \cap R$, in other words, $z \in IS$ and $z \in R$. Write $I = (x_1, \dots, x_n)$, we know that there exists $s_i \in S$ such that $z = \sum s_i x_i$. Now, $z = \phi(z) = \phi\left(\sum s_i x_i\right) = \sum x_i \phi(s_i) \in I$ as desired. ☐

A converse result holds too.

Theorem 2.2. [Hoc77] *Suppose that $R \subseteq S$ is a finite extension of approximately Gorenstein[1] rings. If for every ideal $I \subseteq R$, we have $IS \cap R = S$, then $R \subseteq S$ splits as a map of R-modules.*

Proof. See, [Hoc77] ☐

Consider now what happens if the extension $R \subseteq S$ is the Frobenius map.

Definition 2.3. Given an ideal $I \subseteq R$, the *Frobenius closure of I* (denoted I^F) is the set of all elements $z \in R$ such that $z^{p^e} \in I^{[p^e]}$ for some $e > 0$. Equivalently, it is equal to the set of all elements $z \in R$ such that $z \in (IR^{1/p^e})$ for some ideal I.

Remark 2.4. The set I^F is an ideal. Explicitly, if $z_1, z_2 \in I^F$, then $z_1^{p^a} \in I^{[p^a]}$ and $z_2^{p^b} \in I^{[p^b]}$. Notice that we may assume that $a = b$. Thus $z_1 + z_2 \in I^F$. On the other hand, clearly $hz_1 \in I^F$ for any $h \in R$.

[1]Nearly all rings in geometry satisfy this condition. Explicitly, a local ring $(R, \mathfrak{m})$ is called approximately Gorenstein if for every integer $N > 0$, there exists $I \subseteq \mathfrak{m}^N$ such that R/I is Gorenstein.

We'll point out a couple basic facts about I^F.

Proposition 2.5. *Fix R to be a domain and $(x_1, \ldots, x_n) = I \subseteq R$ an ideal.*

(i) $(I^F)^F = I^F$.

(ii) *For any multiplicative set W, $(W^{-1}I)^F = W^{-1}(I^F)$.*

(iii) *R is F-pure/split if and only if $I = I^F$ for all ideals $I \subseteq R$.*

Proof. For (i), suppose that $z \in (I^F)^F$. Thus there exists an $e > 0$ such that $z^{p^e} \in (I^F)^{[p^e]}$. In particular, we can write $z^{p^e} = \sum a_i x_i^{p^e}$ for some $a_i \in R$ and $x_i \in I^F$. Thus for each x_i, there exists an $e_i > 0$ such that $x_i^{p^{e_i}} \in I^{[p^{e_i}]}$. Choosing $e' \geq e_i$ for all i, we have that $x_i^{p^{e'}} \in I^{[p^{e'}]}$. Therefore, $(z^{p^e})^{p^{e'}} = z^{p^{e+e'}} = \sum a_i^{p^{e'}} x_i^{p^{e+e'}} \in I^{[p^{e+e'}]}$ as desired.

For (ii), we note that $(\supseteq)$ is obvious. Conversely, suppose that $z \in (W^{-1}I)^F$, thus $z^{p^e} \in (W^{-1}I)^{[p^e]} = W^{-1}(I^{[p^e]})$. Therefore, for some $w \in W$, $wz^{p^e} \in I^{[p^e]}$, which implies that $(wz)^{p^e} \in I^{[p^e]}$ and the converse inclusion holds.

Part (iii) is obvious by Theorem 2.2. $\square$

It is natural to hope that these ideas can be extended to (strong) F-regularity.

Recall that R is strongly F-regular (a domain) if for each $0 \neq c \in R$, there exists a map $\phi : F^e_* R \to R$ that sends $c \mapsto 1$ for some $e > 0$.

Definition 2.6. [HH90] Suppose that R is an F-finite domain and I is an ideal of R, then the *tight closure of I* (denoted I^*) is defined to be the set

$$\{z \in R | \exists 0 \neq c \in R \text{ such that } cz^{p^e} \in I^{[p^e]} \text{ for all } e \geq 0\}.$$

Proposition 2.7. *Suppose we have an ideal $(x_1, \ldots, x_n) = I \subseteq R$ where R is an F-finite domain.*

(i) *I^* is an ideal containing I.*

(ii) $(I^*)^* = I^*$.

(iii) *It is known that the formation of I^* does* NOT *commute with localization.*

(iv) *If R is strongly F-regular, then $I^* = I$ for all ideals I.*

Proof. For (i), suppose that $cz^{p^e} \in I^{[p^e]}$ and $dy^{p^e} \in I^{[p^e]}$ for all $e \geq 0$ for certain $c, d \in R \setminus \{0\}$. Then $cd(z+y)^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. Of course, clearly I^* contains I (choose $c = 1$).

For (iv), suppose that $z \in I^*$ and R is strongly F-regular. Choose $c \neq 0$ such that $cz^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. We know that there exists an $e > 0$ and $\phi : F^e_* R \to R$ which sends c to 1. Write $cz^{p^e} = \sum a_i x_i^{p^e}$. Then $z = \phi(cz^{p^e}) = \sum x_i \phi(a_i) \in I$. $\square$

Conjecture 2.8 (Weak$\Rightarrow$Strong)**.** *If $I^* = I$ for all ideals $I \subseteq R$, then R is strongly F-regular.*

Remark 2.9. This conjecture is known for $\mathbb{Q}$-Gorenstein rings (or even local rings which are $\mathbb{Q}$-Gorenstein on the punctured spectrum), for graded rings, and also for rings of finite type over an uncountable field.

References

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