F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 9/21-2010

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1. Pairs in positive characteristic

We've already studied pairs in a certain context. Consider pairs of the form (R, ϕ) where $\phi : F^e_* R \to R$ is an R-linear map. Our first goal will be to see that (R, ϕ) is very like a pair (X, Δ) where $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Proposition 1.1. *Suppose that X is a normal F-finite algebraic variety. Then there is a surjective map from non-zero elements $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ to $\mathbb{Q}$-divisors Δ such that $(p^e - 1)(K_X + \Delta) \sim 0$. Furthermore, two elements ϕ_1, ϕ_2 induce the same divisor if and only if there is a unit $u \in H^0(X, F^e_* \mathcal{O}_X)$ such that $\phi_1(u \cdot \_\_) = \phi_2(\_\_)$.*

More generally, there is a bijection of sets between effective $\mathbb{Q}$-divisors Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index[1] *not divisible by $p > 0$ and certain equivalence relations on pairs $(\mathscr{L}, \phi : F^e_* \mathscr{L} \to \mathcal{O}_X)$ where $\mathscr{L}$ is a line bundle.*

The equivalence relation described above is generated by equivalences of the following two forms.

- *Consider two pairs $(\mathscr{L}_1, \phi_1 : F^{e_1} \mathscr{L}_1 \to \mathcal{O}_X)$ and $(\mathscr{L}_2, \phi_2 : F^{e_2} \mathscr{L}_2 \to \mathcal{O}_X)$ where $e_1 = e_2 = e$. Then we declare these pairs equivalent if there is an isomorphism of line bundles $\psi : \mathscr{L}_1 \to \mathscr{L}_2$ and a commutative diagram:*

$$\begin{array}{ccc} F^e_* \mathscr{L}_1 & \xrightarrow{F^e_* \psi} & \mathbb{F}^e_* \mathscr{L}_2 \\ & {\scriptstyle \phi_1} \searrow \quad \swarrow {\scriptstyle \phi_2} & \\ & \mathcal{O}_X & \end{array}$$

- *Given a pair $(\mathscr{L}, \phi : F^e_* \mathscr{L} \to \mathcal{O}_X)$, we also declare it to be equivalent to the pair $(\mathscr{L}^{p^{(n-1)e} + \dots + 1}, \phi^n : F^{ne} : \mathscr{L}^{p^{(n-1)e + \dots + 1}} \to \dots \to \mathscr{L} \to \mathcal{O}_X)$.*

First we do an example.

Example 1.2. Suppose R is a local ring and $X = \mathrm{Spec}\, R$. Further suppose that R is Gorenstein (or even such that $(p^e - 1)K_X$ is Cartier), then $\mathrm{Hom}_R(F^e_* R, R) \cong F^e_* R$ as we've seen. The generating map $\Phi_R \in \mathrm{Hom}_R(F^e_* R, R)$ corresponds to the zero divisor by the description above. Generally speaking, if $\psi(__) = \Phi_R(x \cdot __)$for $x \in F^e_* R$, then $\Delta_\psi = \frac{1}{p^e - 1} \mathrm{div}_X(x)$. Even without the Gorenstein hypothesis, viewing $\mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta_\phi \rceil), R) \subseteq \mathrm{Hom}_R(F^e_* R, R)$, we have that ϕ generates $\mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta_\phi \rceil), R)$ as an $F^e_* R$-module.

Explicitly, consider $R = k[x]$. We know $\Phi_R : F^e_* R \to R$ is the map that sends $x^{p^e - 1}$ to 1 and the other relevant monomials to zero. Given a general element $\psi : F^e_* R \to R$ defined by

[1] The *index* of a $\mathbb{Q}$-Cartier divisor D is the smallest positive integer n such that $n(K_X + \Delta)$ is Cartier.

the rule

$$\begin{aligned} x^{p^e-1} &\longmapsto a_0 \\ x^{p^e-2} &\longmapsto a_1 \\ \cdots &\longmapsto \cdots \\ x^1 &\longmapsto a_{p^e-2} \\ 1 &\longmapsto a_{p^e-1} \end{aligned}$$

Then $\psi(\_\_) = \Phi_R\left((a_0^{p^e} + a_1^{p^e}x + \cdots + a_{p^e-2}x^{p^e-2} + a_{p^e-1}x^{p^e-1}) \cdot \_\_\right)$ and so $\mathrm{div}_\psi = \frac{1}{p^e-1}\mathrm{div}(a_0^{p^e} + a_1^{p^e}x + \cdots + a_{p^e-2}x^{p^e-2} + a_{p^e-1}x^{p^e-1})$. One can do similarly easy computations for polynomial rings in general.

Now we give a proof of the proposition.

Proof. For the first equivalence, given $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, F^e_*\mathcal{O}_X((1-p^e)K_X))$ define a divisor D_ϕ to be the effective divisor determined by ϕ linearly equivalent to $(1-p^e)K_X$. Set $\Delta_\phi = \frac{1}{p^e-1}D_\phi$. It is easy to see that $(p^e-1)(K_X + \Delta_\phi) \sim 0$.

Now, if ϕ_1 and ϕ_2 induce the same divisor, then $D_{\phi_1} = D_{\phi_2}$ which means that ϕ_1 and ϕ_2 are unit multiples of each other (as sections of $H^0(X, F^e_*\mathcal{O}_X((1-p^e)K_X))$) and the result follows.

For the more general statement, given $\phi : \mathrm{Hom}_{\mathcal{O}_X}(F^e_*\mathscr{L}, \mathcal{O}_X) \cong H^0(X, F^e_*\mathscr{L}^{-1}((1-p^e)K_X))$, we can associate a divisor D_ϕ such that $\mathcal{O}_X(D_\phi) \cong \mathscr{L}^{-1}((1-p^e)K_X)$ and define $\Delta_\phi = \frac{1}{p^e-1}D_\phi$. That the first equivalence relation holds is the same as in the case that $\mathscr{L} = \mathcal{O}_X$ above. The fact that the second equivalence relation holds, is an easy consequence of the following lemma. After the proof of this lemma, it is an easy exercise to verify that these two equivalence relations are all that is needed. ∎

Before doing this lemma, let us do an example.

Lemma 1.3. *Suppose that $\mathscr{L}_1$ and $\mathscr{L}_2$ are line bundles and $\phi_1 : F^{e_1}\mathscr{L}_1 \to \mathcal{O}_X$ and $\phi_2 : F^{e_2}\mathscr{L}_2 \to \mathcal{O}_X$ are $\mathcal{O}_X$-linear maps. We can then define a composition of these maps as follows: Consider $\psi := \phi_2 \circ (F^{e_2}_*(\mathscr{L}_2 \otimes \phi_1)) : F^{e_1+e_2}\mathscr{L}_1 \otimes \mathscr{L}_2^{p^{e_1}} \to \mathscr{L}_2$. Then*

$$\Delta_\psi = \frac{p^{e_1}-1}{p^{e_1+e_2}-1}\Delta_{\phi_1} + \frac{p^{e_1}(p^{e_2}-1)}{p^{e_1+e_2}-1}\Delta_{\phi_2}$$

Notice that $\frac{p^{e_1}-1}{p^{e_1+e_2}-1} + \frac{p^{e_1}(p^{e_2}-1)}{p^{e_1+e_2}-1} = 1$.

Proof. The statement is local, so we may assume that $\mathscr{L}_1 \cong \mathscr{L}_2 \cong \mathcal{O}_X$. In fact, we may assume that X is the prime spectrum of a DVR R with parameter r. Fix $\Psi_R : F_*R \to R$ to be the generating map of $\mathrm{Hom}_R(F_*R, R)$ as an F_*R-module.

In this case, $\phi_1(\_\_) = \Psi^{e_1}_R(x_1 \cdot \_\_)$ and $\phi_2(\_\_) = \Psi^{e_2}_R(x_2 \cdot \_\_)$ where $x_i \in F^{e_i}_*R$ and so $\Delta_{\phi_i} = \frac{1}{p^{e_i}-1}\mathrm{div}_X(x_i)$. Then

$$\phi_2(F^{e_2}_*\phi_1(\_\_)) = \Psi^{e_2}_R\left(F^{e_2}_*x_2\Psi^{e_1}_R\left(F^{e_1}_*x_1\_\_\right)\right) = \Psi^{e_1+e_2}\left(F^{e_1+e_2}_*x_1x_2^{p^{e_1}}\_\_\right)$$

The divisor of this composition is evidently

$$\frac{1}{p^{e_1+e_2}-1}(\mathrm{div}(x_1)+p^{e_1}\mathrm{div}(x_2)) = \frac{1}{p^{e_1+e_2}-1}(\mathrm{div}(x_1)+p^{e_1}\mathrm{div}(x_2)) = \frac{p^{e_1}-1}{p^{e_1+e_2}-1}\Delta_{\phi_1} + \frac{p^{e_1}(p^{e_2}-1)}{p^{e_1+e_2}-1}\Delta_{\phi_2}$$

□

Lemma 1.4. *An element $\phi \in \mathrm{Hom}_R(F^e_* R, R)$ is contained inside the submodule*

$$\mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R) \subseteq \mathrm{Hom}_R(F^e_* R, R) \tag{1}$$

if and only if $D_\phi \geq (p^e - 1)\Delta$.

Proof. Because all the module are reflexive the statement can be reduced to the case when R is a discrete valuation ring and $\Delta = s\,\mathrm{div}(x)$ where x is the parameter for the DVR R and $s \geq 0$ is a real number. In this case, the inclusion from equation 1 can be identified with the multiplication map $R \to R$ which sends 1 to $x^{\lceil s(p^e-1) \rceil}$. Thus, $\phi \in \mathrm{Hom}_R(F^e_* R, R) \cong R$ is contained inside $\mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R) \cong x^{\lceil s(p^e-1) \rceil} R$ if and only if $D_\phi \geq \lceil s(p^e - 1) \rceil \,\mathrm{div}(x) = \lceil (p^e - 1)\Delta \rceil$. However, since D_ϕ is integral, it is harmless to remove the round-up $\lceil \cdot \rceil$. □

Remark 1.5. One can work with non-effective divisors similarly. One then can consider maps $\phi : F^e_* \mathscr{L} \to K(X)$ where $K(X)$ is the fraction field of X.

Definition 1.6. [HH89], [HW02], [Tak04], [Sch08] Suppose that $(X, \Delta, \mathfrak{a}^t)$ is a triple where X is an F-finite normal scheme, Δ is an effective $\mathbb{Q}$-divisor, $\mathfrak{a}$ is an ideal sheaf and $t \geq 0$ is a real number. Further suppose that X is the spectrum of a local ring R. We say that $(X, \Delta, \mathfrak{a}^t)$ is:

(a) *sharply F-pure* if there exists some $e > 0$ and some $\phi \in \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$ such that $1 \in \phi(F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil})$.
(b) *strongly F-regular* if for every $c \in R \setminus 0$, there exists a $e > 0$ and some $\phi \in \mathrm{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta \rceil), R)$ such that $1 \in \phi(F^e_* c\mathfrak{a}^{\lceil t(p^e-1) \rceil})$.

If X is not the spectrum of a local ring, then we generalize these definitions by requiring them at every point. They are open conditions.

Suppose that X is quasi-projective. The *(big) test ideal of* $(X, \Delta, \mathfrak{a}^t)$, denoted $\tau_b(X, \Delta, \mathfrak{a}^t)$ is defined to be the unique smallest non-zero ideal of $J \subseteq R$ such that $\phi(F^e_* \mathfrak{a}^{\lceil t(p^e-1) \rceil} J \mathscr{L}) \subseteq J$ for every $\phi : F^e_* \mathscr{L} \to \mathcal{O}_X$ such that $\Delta_\phi \geq \Delta$. This always exists and its formation commutes with localization.

Definition 1.7. We say that R is *strongly F-regular / F-pure* if the same statement holds for $\Delta = 0$ and $\mathfrak{a} = R$.

Remark 1.8. If $\Delta = \Delta_\psi$ for some $\psi : F^e_* \mathscr{L} \to \mathcal{O}_X$, then in the definition of the big test ideal / sharp F-purity / strong F-regularity, one only needs to check the condition for $\phi = \psi^n$.

Proposition 1.9. *A ring is strongly F-regular if and only if $\tau_b(R) = R$. Furthermore a strongly F-regular ring is always F-rational (in particular, it is Cohen-Macaulay) and a Gorenstein F-rational ring is strongly F-regular.*

Proof. Suppose R is strongly F-regular and local and suppose that J satisfies $\phi(F^e_* J) \subseteq J$ for every $\phi : F^e_* R \to R$. The strong F-regularity hypothesis implies immediately that J contains R and is thus equal to 1. Conversely, suppose that $\tau_b(R) \neq R$, then choose any element $0 \neq c \in \tau_b(R)$. It follows that for every $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, $\phi(F^e_* cR) \subseteq \tau_b(R)$ which does not contain 1.

We've already seen that F-rational Gorenstein rings are strongly F-regular. This is simply because $R = \omega_R$ and in this case we have a map $\Psi_R : F^e_*(\omega_R = R) \to (\omega_R = R)$ such that

$\tau(R, \Psi_R) = R$ (interestingly, we don't need the Cohen-Macaulay condition here, it is implied for free by what follows).

Now assume that R is strongly F-regular, we will show it is F-rational and in particular Cohen-Macaulay (this is one proof where I think it is easier to use the tight closure definitions). First note that R is necessarily normal since we know the conductor is ϕ-compatible for all $\phi \in \mathrm{Hom}_R(F^e_* R, R)$, so in particular $\tau_b(R)$ is always contained in the conductor. We now show that R is Cohen-Macaulay by showing that $h^i(\omega^\bullet_R) = 0$ for all $i > -\dim R$. Suppose not, so choose $0 \neq c \in R$ such that $ch^i(\omega^\bullet_R) = 0$ but $h^i(\omega^\bullet_R) \neq 0$ (such modules always have support strictly smaller in dimension than the ring so this is possible). Dual to the map $R \to F^e_* R$ which sends $1 \mapsto c$, we have the map

$$h^i(F^e_* \omega^\bullet_R) \xrightarrow{F^e_* \times c} h^i(F^e_* \omega^\bullet_R) \longrightarrow h^i(\omega^\bullet_R)$$

For e large enough, this map is necessarily surjective (since our map $R \to F^e_* R$ splits), but this is ridiculous since it is also zero.

Using the same argument, we also have that the composition $F^e_* c\omega_R \subseteq F^e_* \omega_R \to \omega_R$ surjects for all $e \gg 0$. But this clearly implies that $\tau(\omega_R) = \omega_R$ which completes the proof. □

Remark 1.10. We have the following implications:

$$\begin{array}{ccc} \text{strongly } F\text{-regular} & \Longrightarrow & F\text{-rational} \\ \Downarrow & & \Downarrow \\ F\text{-pure} & \Longrightarrow & F\text{-injective} \end{array}$$

Furthermore, under the (quasi)-Gorenstein hypothesis the horizontal arrows can be reversed.

Proposition 1.11. *The ideal* $\tau_b(X, \Delta, \mathfrak{a}^t)$ *exists.*

Proof. I'll only prove this for $X = \operatorname{Spec} R$. Choose a non-zero $\psi \in M^e_{\Delta,\mathfrak{a}^t} = (F^e_* \mathfrak{a}^{\lceil t(p^e-1)\rceil}) \cdot \mathrm{Hom}_R(F^e_* R(\lceil (p^e-1)\Delta\rceil), R)$. We view ψ as a map from $F^e_* R$ to R. Choose c a test element for the pair (R, ψ). Then we claim that

$$\tau_b(R, \Delta, \mathfrak{a}^t) = \sum_{e \geq 0} \sum_{\phi \in M^e_{\Delta,\mathfrak{a}^t}} \phi(F^e_* cR).$$

It is enough to show equality after localizing at each prime ideal, and so we may assume R is local. The sum is is stabilized by all $\phi \in M^e_{\Delta,\mathfrak{a}^t}$. There is a computation here to check this, that the elements of $M^e_{\Delta,\mathfrak{a}^t}$ form an algebra of maps, but it is of the form $p^d\lceil (p^e-1)t\rceil + \lceil (p^d-1)t\rceil \geq \lceil p^{e+d}-1\rceil$. On the other hand, clearly any $J \subseteq R$ that is stabilized by all $\phi \in M^e_{\Delta,\mathfrak{a}^t}$ contains c since all powers of ψ live in $M^e_{\Delta,\mathfrak{a}^t}$ for various e. □

2. F-singularities and birational maps

Our goal in this section is to relate F-singularities and test ideals with log canonical and log terminal singularities as well as multiplier ideals. In order to do this, we need to explain how maps $\phi : F^e_* R \to R$ behave under birational maps.

Proposition 2.1. *Suppose that* $\pi : \widetilde{X} \to X$ *is a proper birational map and* $\phi \in \mathrm{Hom}_R(F^e_* R, R)$. *Write*

$$K_{\widetilde{X}} - \sum a_i E_i = f^*(K_R + \Delta_\phi)$$

Then ϕ induces a map $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)\sum a_i E_i) \to \mathcal{O}_{\widetilde{X}}$ which agrees with ϕ where π is an isomorphism. Finally, it induces a map (which we also call $\widetilde{\phi}$)

$$\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil).$$

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