F-SINGULARITIES AND FROBENIUS SPLITTING NOTES
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Remark 0.1. If Δ is effective, we see that $(X, \Delta, \mathfrak{a}^t)$ is klt if and only if $\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \mathcal{O}_X$. Furthermore, if $(X, \Delta, \mathfrak{a}^t)$ is log canonical, then $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ is a radical ideal. Furthermore, if $(X, \Delta, \mathfrak{a}^t)$ is klt and $\Delta \geq 0$, then $\lfloor \Delta \rfloor = 0$.

Example 0.2. Consider $X = \mathbb{A}^2$ and $\Delta = \frac{2}{3}\operatorname{div}_X(xy(x-y))$. A log resolution $\pi : \widetilde{X} \to X$ can be obtained by doing one blow-up at the origin, use E to denote the exceptional divisor. We set $K_X = 0$, then

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) = K_{\widetilde{X}} - \frac{2}{3}\operatorname{div}_{\widetilde{X}}(xy(x-y)) = E - \frac{2}{3}(3E + C_1 + C_2 + C_3) = -E - \frac{2}{3}(C_1 + C_2 + C_3)$$

where the C_i are the strict transforms of the three curves in the support of Δ. Thus (X, Δ) is log canonical, but not Kawamata/purely log terminal. Furthermore, $\mathcal{J}(X, \Delta) = (x, y) = \mathfrak{m}$.

An example of a plt pair that is not klt is $(\mathbb{A}^2, \operatorname{div}(x))$. Generally speaking the pair made up of a smooth variety and a smooth divisor is always purely log terminal, but a pair made up of a smooth variety and a simple normal crossings divisor is not plt – $(\mathbb{A}^2, \operatorname{div}(xy))$ is not purely log terminal (even though it is its own log resolution).

In general, klt singularities are rational, klt singularities are log canonical, Gorenstein rational singularities are klt. Log canonical singularities are Du Bois and Gorenstein Du Bois singularities are log canonical.

Proposition 0.3. [Elk81] *If (X, Δ) is klt and $\Delta \geq 0$, then X has rational singularities. If X is Gorenstein, then if X has rational singularities, X has canonical (and thus klt) singularities.*

Proof. Let $\pi : \widetilde{X} \to X$ be a log resolution. We have a natural inclusion $\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta)\rceil)$. Applying $R\pi_*$ gives us the composition

$$\mathcal{O}_X \to R\pi_*\mathcal{O}_{\widetilde{X}} \to R\pi_*\mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG\rceil) \cong \mathcal{J}(X, \Delta) = \mathcal{O}_X$$

This map is clearly an isomorphism in codimension 1, and so it is an isomorphism. Thus $\mathcal{O}_X \to R\pi_*\mathcal{O}_{\widetilde{X}}$ splits, and so X has rational singularities.

In the Gorenstein case, for the converse direction, if $\omega_X \cong R\pi_*\omega_{\widetilde{X}}$, then $\mathcal{O}_X \cong R\pi_*\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}} - \pi^*K_X)$. ☐

Proposition 0.4. [KK09] *If (X, Δ) is log canonical, then X has Du Bois singularities.*

Proof. We only provide a proof in the Cohen-Macaulay case (which is the only case where we defined Du Bois singularities). Set $\pi : \widetilde{X} \to X$ to be a log resolution with reduced exceptional divisor E. There exists a natural inclusion $\iota : \varrho_*\omega_{X'}(G) \subseteq \omega_X$, so the question is local. We may assume that X is affine and need only prove that every section of ω_X is already contained in $\varrho_*\omega_{X'}(G)$.

Next, choose a canonical divisor $K_{X'}$ and let $K_X = \varrho_* K_{X'}$. As $\Delta' = \varrho_*^{-1}\Delta$, it follows that the divisors $K_{X'} + \Delta'$ and $\varrho_*^{-1}(K_X + \Delta) = \Delta'$ may only differ in exceptional components. We emphasize that these are actual divisors, not just equivalence classes (and so are B and B').

Since X and X' are birationally equivalent, their function fields are isomorphic. Let us identify $K(X)$ and $K(X')$ via ρ^* and denote them by K. Further let $\mathscr{K}$ and $\mathscr{K}'$ denote the K-constant sheaves on X and X' respectively.

Now we have the following inclusions:

$$\Gamma(X, \varrho_* \omega_{X'}(E)) \subseteq \Gamma(X, \omega_X) \subseteq \Gamma(X, \mathscr{K}) = K,$$

and we need to prove that the first inclusion is actually an equality. Let $g \in \Gamma(X, \omega_X)$. So

$$0 \leq \mathrm{div}_X(g) + K_X \leq \mathrm{div}_X(g) + K_X + \Delta \tag{1}$$

As (X, Δ) is log canonical, there exists an $m \in \mathbb{N}$ such that $mK_X + m\Delta$ is a Cartier divisor and hence can be pulled back to a Cartier divisor on X'. By the choices we made earlier, we have that $\varrho^*(mK_X + m\Delta) = mK_{X'} + m\Delta' + \Theta$ where Θ is an exceptional divisor.

However, using the fact that (X, Δ) is log canonical, one obtains that $\Theta \leq mG$. Combining this with (1) gives that

$$0 \leq \mathrm{div}_{X'}(g^m) + \varrho^*(mK_X + m\Delta) \leq m\big(\mathrm{div}_{X'}(g) + K_{X'} + \Delta' + G\big),$$

and in particular we obtain that

$$\mathrm{div}_{X'}(g) + K_{X'} + \Delta' + G \geq 0.$$

We claim that:

$$\mathrm{div}_{X'}(g) + K_{X'} + G \geq 0.$$

Proof. By construction

$$\mathrm{div}_{X'}(g) + K_{X'} + G = \varrho_*^{-1}\underbrace{(\mathrm{div}_X(g) + K_X)}_{\geq 0} + \underbrace{F + G}_{\text{exceptional}}\ . \tag{2}$$

Where F is an appropriate exceptional divisor, though it is not necessarily effective. We also have that

$$\mathrm{div}_{X'}(g) + K_{X'} + G = \underbrace{\mathrm{div}_{X'}(g) + K_{X'} + \Delta' + G}_{\geq 0} - \underbrace{D'}_{\text{non-exceptional}}\ . \tag{3}$$

Now let A be an arbitrary irreducible component of $\mathrm{div}_{X'}(g) + K_{X'} + G$. If A were not effective, it would have to be exceptional by (2) and non-exceptional by (3). Hence A must be effective and the claim is proven. □

It follows that $g \in \Gamma(X', \omega_{X'}(G)) = \Gamma(X, \varrho_* \omega_{X'}(G))$, completing the proof. □

0.1. **The log terminal and log canonical conditions for cones.** We study the condition that (Y, Δ_Y) has log canonical/terminal singularities when $Y = \operatorname{Spec} S$ is the affine cone over a projective variety X and Δ_Y corresponds to the pull-back of some $\mathbb{Q}$-divisor Δ_X on X via the k^*-bundle $Y \setminus V(S_+) \to X$ (or rather the closure of the pullback).

Suppose that (X, Δ_X) is a log $\mathbb{Q}$-Gorenstein pair and that A is an ample divisor. Set $S = \oplus H^0(X, \mathcal{O}_X(nA))$ to be the section ring and $Y = \operatorname{Spec} S$ and Δ_Y as above.

Proposition 0.5. *The pair (Y, Δ_Y) is klt (respectively lc) if and only if (X, Δ_X) is klt (respectively lc) and $-(K_X + \Delta_X) = rA$ for some $r \in \mathbb{Q}_{>0}$ (respectively $r \in \mathbb{Q}_{\geq 0}$).*

Remark 0.6. This proposition says that (X, Δ_X) is log Fano if and only if (Y, Δ_Y) is klt for some section ring. Likewise, (X, Δ) is log Calabi-Yau is equivalent to the condition that (Y, Δ_Y) is lc with lc-center at the origin.

Proof. Certainly the fact that (X, Δ_X) is klt/lc is necessary because of the k^*-bundle description of $Y \setminus V(S_+) \to X$ described above. For simplicity we assume now that A is (very (very)) ample. We can reduce to this case using Veronese cover tricks which I won't describe here.

First we ask ourselves what it means that $(K_Y + \Delta_Y)$ is $\mathbb{Q}$-Cartier (recall, that K_Y is just the sheaf associated to K_X via pull-back). This means that $n(K_Y + \Delta_Y)$ is locally free, and because we are working in the graded setting, this just means that $\mathcal{O}_Y(n(K_Y + \Delta_Y)) = \mathcal{O}_Y(m)$. But this is equivalent to the requirement that $n(K_X + \Delta_X) \sim mA$.

We now blow-up to origin of Y giving us a map $\pi : \widetilde{Y} \to Y$. There is one exceptional divisor E of this map and E is isomorphic to X. Furthermore, restricting $\mathcal{O}_{\widetilde{Y}}(-E)$ to E yields $\mathcal{O}_X(A)$.

Write $K_{\widetilde{Y}} - \pi^*(K_Y + \Delta_Y) = aE - \pi_*^{-1}\Delta_Y$. It is clear that $\pi_*^{-1}\Delta_Y|_E = \Delta_X$. However, we also know that $(K_{\widetilde{Y}} + E)|_E = K_X$. Rewriting our first equation gives us $\pi^*(K_Y + \Delta_Y) = K_{\widetilde{Y}} - aE + \pi_*^{-1}\Delta_Y$. Therefore

$$0 \sim (K_{\widetilde{Y}} + E - (a+1)E + \pi_*^{-1}\Delta_Y)|_E = K_X + (a+1)A + \Delta_Y$$

or in other words, $-(K_X + \Delta_Y) \sim (a+1)A$. In particular, if (Y, Δ) klt (respectively lc) then $a > 0$ (respectively $a \geq 0$). Thus $-(K_X + \Delta_Y)$ is some positive rational multiple of A (respectively, $-(K_X + \Delta_Y)$ is some non-negative multiple of A).

Conversely, if $-(K_X + \Delta_Y)$ is some positive rational multiple of A and (X, Δ_X) is klt, it can be shown that (Y, Δ_Y) is klt. We will not do this now though. There are two approaches, the most direct is to do a complete resolution of singularities followed by some analysis. The second is to use inversion of adjunction which allows one to relate the singularities of a divisor with the singularities of a pair. We'll cover more on this second topic later. $\square$

1. PAIRS IN POSITIVE CHARACTERISTIC

We've already studied pairs in a certain context. Consider pairs of the form (R, ϕ) where $\phi : F^e_* R \to R$ is an R-linear map. Our first goal will be to see that (R, ϕ) is very like a pair (X, Δ) where $K_X + \Delta$ is $\mathbb{Q}$-Cartier.

Proposition 1.1. *Suppose that X is a normal F-finite algebraic variety. Then there is a surjective map from non-zero elements $\phi \in \mathrm{Hom}_{\mathcal{O}_X}(F^e_* \mathcal{O}_X, \mathcal{O}_X)$ to $\mathbb{Q}$-divisors Δ such that $(p^e - 1)(K_X + \Delta) \sim 0$. Furthermore, two elements ϕ_1, ϕ_2 induce the same divisor if and only if there is a unit $u \in H^0(X, F^e_* \mathcal{O}_X)$ such that $\phi_1(u \cdot \_\_) = \phi_2(\_\_)$.*

More generally, there is a bijection of sets between effective $\mathbb{Q}$-divisors Δ such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier with index[1] not divisible by $p > 0$ and certain equivalence relations on pairs $(\mathscr{L}, \phi : F^e_* \mathscr{L} \to \mathcal{O}_X)$ where $\mathscr{L}$ is a line bundle.

The equivalence relation described above is generated by equivalences of the following two forms.

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[1] The *index* of a $\mathbb{Q}$-Cartier divisor D is the smallest positive integer n such that $n(K_X + \Delta)$ is Cartier.

- *Consider two pairs* $(\mathscr{L}_1, \phi_1 : F^{e_1}\mathscr{L}_1 \to \mathcal{O}_X)$ *and* $(\mathscr{L}_2, \phi_2 : F^{e_2}\mathscr{L}_2 \to \mathcal{O}_X)$ *where* $e_1 = e_2 = e$*. Then we declare these pairs equivalent if there is an isomorphism of line bundles* $\psi : \mathscr{L}_1 \to \mathscr{L}_2$ *and a commutative diagram:*

$$\begin{tikzcd}
F^e_*\mathscr{L}_1 \arrow[rr, "F^e_*\psi"] \arrow[dr, "\phi_1"'] & & \mathbb{F}^e_*\mathscr{L}_2 \arrow[dl, "\phi_2"] \\
& \mathcal{O}_X &
\end{tikzcd}$$

- *Given a pair* $(\mathscr{L}, \phi : F^e_*\mathscr{L} \to \mathcal{O}_X)$*, we also declare it to be equivalent to the pair* $(\mathscr{L}^{p^{(n-1)e}+\dots+1}, \phi^n : F^{ne} : \mathscr{L}^{p^{(n-1)e+\dots+1}} \to \dots \to \mathscr{L} \to \mathcal{O}_X)$*.*

First we do an example.

Example 1.2. Suppose R is a local ring and $X = \operatorname{Spec} R$. Further suppose that R is Gorenstein (or even such that $(p^e - 1)K_X$ is Cartier), then $\operatorname{Hom}_R(F^e_*R, R) \cong F^e_*R$ as we've seen. The generating map $\Phi_R \in \operatorname{Hom}_R(F^e_*R, R)$ corresponds to the zero divisor by the description above. Generally speaking, if $\psi(__) = \Phi_R(x \cdot __)$for $x \in F^e_*R$, then $\Delta_\psi = \frac{1}{p^e-1}\operatorname{div}_X(x)$. Even without the Gorenstein hypothesis, viewing $\operatorname{Hom}_R(F^e_*R(\lceil (p^e-1)\Delta_\phi \rceil), R) \subseteq \operatorname{Hom}_R(F^e_*R, R)$, we have that ϕ generates $\operatorname{Hom}_R(F^e_*R(\lceil (p^e-1)\Delta_\phi \rceil), R)$ as an F^e_*R-module.

Explicitly, consider $R = k[x]$. We know $\Phi_R : F^e_*R \to R$ is the map that sends x^{p^e-1} to 1 and the other relevant monomials to zero. Given a general element $\psi : F^e_*R \to R$ defined by the rule

$$\begin{tikzcd}
x^{p^e-1} \arrow[r, mapsto] & a_0 \\
x^{p^e-2} \arrow[r, mapsto] & a_1 \\
\cdots \arrow[r, mapsto] & \cdots \\
x^1 \arrow[r, mapsto] & a_{p^e-2} \\
1 \arrow[r, mapsto] & a_{p^e-1}
\end{tikzcd}$$

Then $\psi(\_\_) = \Phi_R\left((a_0^{p^e} + a_1^{p^e}x + \dots + a_{p^e-2}x^{p^e-2} + a_{p^e-1}x^{p^e-1}) \cdot \_\_\right)$ and so $\operatorname{div}_\psi = \frac{1}{p^e-1}\operatorname{div}(a_0^{p^e} + a_1^{p^e}x + \dots + a_{p^e-2}x^{p^e-2} + a_{p^e-1}x^{p^e-1})$. One can do similarly easy computations for polynomial rings in general.

REFERENCES

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