F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 10/28-2010

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1. Characteristic $p > 0$ analogs of LC-centers and subadjunction continued

Using the same idea (Fedder's lemma), we have the following method for checking whether an ideal is a non-F-regular center.

Proposition 1.1. *Suppose that S is a regular F-finite ring and that $R = S/I$. Suppose that $Q \in \operatorname{Spec} S$ contains I. Then Q/I is a non-F-regular center of R if and only if $I^{[p^e]} : I \subseteq Q^{[p^e]} : Q$ for all $e \geq 0$. Furthermore, if R/I is $\mathbb{Q}$-Gorenstein such that $(p^e - 1)K_R$ is Cartier, then one may check that single $e > 0$.*

On the other hand if S is sufficiently local and if a $\mathbb{Q}$-divisor Δ on $\operatorname{Spec} R$ corresponds to a map $\phi : F^e_* R \to R$. Fix $d \in S \in I^{[p^e]} : I$ corresponding to ϕ. Then Q/I is a non-F-regular center of (R, Δ) if and only if $d \in Q^{[p^e]} : Q$.

Proof. The statements are local, so we may assume that S is local. But then the result follows immediately one recalls that $F^e_*(I^{[p^e]} : I)$ maps surjectively onto $\operatorname{Hom}_R(F^e_* R, R)$. $\square$

Remark 1.2. For a log canonical pair (X, Δ), the set of LC-centers satisfy many remarkable properties. For example, if the pair (X, Δ) is log canonical:

- Any union of such centers is seminormal (and in fact, Du Bois).
- Any intersection of such centers is a union of such centers.

F-pure centers satisfy the analogous results.

One can certainly ask if other natural properties of LC-centers hold for F-pure centers. The set of LC-centers are finite for a log canonical pair, so we can ask the following.

Theorem 1.3. [Sch09], [MK09] *If (X, Δ) is sharply F-pure, then there are finitely many F-pure centers.*

Proof. Choose ϕ such that $\Delta_\phi \geq \Delta$ and that $\phi(1) = 1$. Note, every center of sharp F-purity $Q \in \operatorname{Spec} R$ for (R, Δ) satisfies $\phi(F^e_* Q) \subseteq Q$. We will show that there are finitely many prime ideals Q such that $\phi(F^e_* Q) \subseteq Q$. First note that if there are infinitely many such prime ideals, one can find a collection $\mathfrak{Q}$ of infinitely many centers which all have the same height and whose closure (in the Zariski topology) is an irreducible subscheme of $\operatorname{Spec} R$ with generic point P (ie, P is the minimal associated prime of $\bigcap_{Q \in \mathfrak{Q}} Q$). Notice P must have smaller height than the elements of $\mathfrak{Q}$. Notice further that P also satisfies $\phi(F^e_* P) \subseteq P$ since it is the intersection of the elements of $\mathfrak{Q}$ (in other words, P is a center of sharp F-purity for (R, Δ_ϕ)).

By restricting to an open set, we may assume that R/P is normal (the elements of $\mathfrak{Q}$ will still form a dense subset of $V(P)$). Then ϕ induces a divisor Δ_P on $\operatorname{Spec} R/P$ as above. But the set of elements in $\mathfrak{Q}$ restrict to centers of sharp F-purity for $(R/P, \Delta_P)$ by F-adjunction.

As noted above, $\{Q/P \mid Q \in \mathfrak{Q}\}$ is dense in $\operatorname{Spec} R/P$ and simultaneously $\{Q/P | Q \in \mathfrak{Q}\}$ is contained in the non-strongly F-regular locus of $(R/P, \Delta_P)$, which is closed and proper. This is a contradiction. $\square$

2. F-rationality via alterations and finite maps

In this section, we will show that F-rationality can also be described via alterations. First we prove the *equational lemma*, which lets us kill cohomology in characteristic $p > 0$ by passing to finite covers, the variant we give appeared recently in the work of Huneke-Lyubeznik, but the result has connections to the work of Hochster and others even in the 70s, as well as the work of Hochster-Huneke and Smith.

Theorem 2.1 (Equational-Lemma)**.** [HL07], [HH92] *Let R be a commutative Noetherian domain containing a field of characteristic $p > 0$. Let K be the fraction field of R and suppose that $\bar{K}$ is the algebraic closure of K. Let I be an ideal of R and suppose that $\alpha \in H^i_I(R)$ is an element such that $\alpha, \alpha^p, \alpha^{p^2}, \dots$ belong to a finitely generated R-submodule of $H^i_I(R)$. Then there exists an R-subalgebra R' of $\bar{K}$ that is a finite R-module and such that the induced map $H^i(R) \to H^i(R')$ sends α to zero.*

This proof is taken from [HL07]. Let A_t denote the submodule generated by $\alpha, \alpha^p, \dots, \alpha^{p^t}$. By hypothesis, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ eventually stabilizes at A_s (note we may take s not divisible by p). Thus we have an equation:

$$g(T) = T^{p^s} - r_1 T^{p^{s-1}} - r_2 T^{p^{s-2}} - \dots - r_{s-1} T$$

where $r_i \in R$ and for which α is a root. It is a key point here that g is additive in T (because of the pth powers).

Suppose that $x_1, \dots, x_n$ generate I and consider the Čech complex

$$0 \longrightarrow C^0(M) \xrightarrow{d_0} C^1(M) \xrightarrow{d_1} C^2(M) \longrightarrow \cdots \xrightarrow{d_{n-1}} C^n(M) \longrightarrow 0$$

where $C^0(M) = M$ and $C^i(M) = \oplus_{j_1 \leq \dots \leq j_i} M_{x_{j_1} \dots x_{j_i}}$ (we will set $M = R$ and also equal to certain finite extensions of R).

Suppose that $\widetilde{\alpha} \in C^i(R)$ is a cycle that represents α. We know that $g(\widetilde{\alpha}) = d_{i-1}(\beta) \in d_{i-1}(C^{i-1}(R))$ since $g(\alpha) = 0$. Write

$$\beta = \oplus_{j_1 \leq \dots \leq j_{i-1}} \left(\frac{r_{j_1 \dots j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} \right)$$

for some (uniform) integer e.

For each tuple $j_1 \leq \dots \leq j_i$, consider the equation

$$g\left(\frac{Z_{j_1 \leq \dots \leq j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e}\right) - \frac{r_{j_1 \dots j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} = 0$$

in the variable $Z_{j_1 \leq \dots \leq j_{i-1}}$. Clearing denominators gives us monic polynomials $h_{j_1 \leq \dots \leq j_{i-1}}$ in the variables $Z_{j_1 \leq \dots \leq j_{i-1}}$. Let $z_{j_1 \leq \dots \leq j_{i-1}} \in \bar{K}$ be a root of this equation. Set R'' to be the finite extension of R generated by all the $z_{j_1 \leq \dots \leq j_{i-1}}$.

Set

$$\widetilde{\widetilde{\alpha}} = \oplus \left(\frac{z_{j_1 \leq \dots \leq j_{i-1}}}{(x_{j_1} \dots x_{j_{i-1}})^e} \right) \in C^{i-1}(R'')$$

We also know that $C^\bullet(R)$ is a subcomplex of $C^\bullet(R'')$ and so we can identify $\widetilde{\alpha}$ and β with their natural images in $C^\bullet(R'')$. Thus $\widetilde{\alpha} \in C^i(R'')$ is a cycle representing the image of α under $H^i_I(R) \to H^i_I(R'')$. As is $\bar{\alpha} = \widetilde{\alpha} - d_{i-1}(\widetilde{\widetilde{\alpha}})$ (we just subtracted a boundary, which does not change the cohomology class). Now, $g(\widetilde{\widetilde{\alpha}}) = \beta$ and also $g(\widetilde{\alpha}) = d_{i-1}(\beta)$, so that

$$g(\bar{\alpha}) = g(\widetilde{\alpha} - d_{i-1}(\widetilde{\widetilde{\alpha}})) = g(\widetilde{\alpha}) - g(d_{i-1}(\widetilde{\widetilde{\alpha}})) = g(\widetilde{\alpha}) - d_{i-1}(g(\widetilde{\widetilde{\alpha}})) = d_{i-1}(\beta) - d_{i-1}(\beta) = 0.$$

Write

$$\bar{\alpha} = \oplus \rho_{j_1 \leq \cdots \leq j_i} \text{ with } \rho_{j_1 \leq \cdots \leq j_i} \in R''_{j_1 \dots j_i}.$$

We know that $g(\rho_{j_1 \leq \cdots \leq j_i}) = 0$ individually so that $\rho_{j_1 \leq \cdots \leq j_i}$ is integral over R. Set R' to be R'' adjoin the $\rho_{j_1 \leq \cdots \leq j_i}$ (this is contained in the normalization of R'').

By hypothesis, the image of α in $H^i_I(R')$ is represented by $\bar{\alpha} = \oplus \rho_{j_1 \leq \cdots \leq j_i}$. We need to show that this is a boundary. However, there is an exact subcomplex of $C^\bullet(R')$ which is simply R' in each term, $\bar{\alpha}$ is certainly in this subcomplex and thus it is a boundary as desired. □

References

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