F-SINGULARITIES AND FROBENIUS SPLITTING NOTES 10/19-2010

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1. F-singularities and birational maps

Our goal in this section is to relate F-singularities and test ideals with log canonical and log terminal singularities as well as multiplier ideals. In order to do this, we need to explain how maps $\phi : F^e_* R \to R$ behave under birational maps.

Proposition 1.1. *Suppose that $\pi : \widetilde{X} \to X$ is a proper birational map and $\phi \in \operatorname{Hom}_R(F^e_* R, R)$. Write*

$$K_{\widetilde{X}} - \sum a_i E_i = f^*(K_R + \Delta_\phi)$$

Then ϕ induces a map $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)\sum a_i E_i) \to \mathcal{O}_{\widetilde{X}}$ which agrees with ϕ where π is an isomorphism. Finally, it induces a map (which we also call $\widetilde{\phi}$)

$$\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil).$$

Proof. Throughout, we remove the singular locus of $\widetilde{X}$ if necessary so that it is regular, and work with divisors on this locus. This is harmless though since we are looking at maps between reflexive modules.

By assumption ϕ generates $\operatorname{Hom}_R(F^e_* R(\lceil (p^e - 1)\Delta_\phi \rceil), R) \cong F^e_* R((1-p^e)(K_R + \Delta_\phi)) \cong F^e_* R$. Thus we have a section $d \in f^* R((1-p^e)(K_R + \Delta_\phi)) \cong \mathcal{O}_{\widetilde{X}}$ corresponding to ϕ, and furthermore this section generates. So that we obtain a section $d \in \Gamma(\widetilde{X}, \mathcal{O}_{\widetilde{X}}((1-p^e)(K_{\widetilde{X}} - \sum a_i E_i))$ which generates as an $\mathcal{O}_{\widetilde{X}}$-module. However, $F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)(K_{\widetilde{X}} - \sum a_i E_i)) = \mathscr{H}om_{\mathcal{O}_X}(F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)(\sum a_i E_i)), \mathcal{O}_{\widetilde{X}})$ and we obtain our first statement easily.

For the second statement, consider $\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)\sum a_i E_i) \to \mathcal{O}_{\widetilde{X}}$. Twisting by $\mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil)$ gives us a map

$$\widetilde{\phi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^e)\sum a_i E_i + p^e \lceil \sum a_i E_i \rceil) \to \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil)$$

However, $(1-p^e)\sum a_i E_i + p^e \lceil \sum a_i E_i \rceil \geq \lceil (1-p^e)\sum a_i E_i + p^e \sum a_i E_i \rceil = \lceil \sum a_i E_i \rceil$ which gives the desired map via composition with the inclusion. □

Remark 1.2. Restrict the above map $\widetilde{\phi}$ to an E_i such that $a_i \leq 0$. Localizing at the generic point of that E_i gives us a "generating" map from $\mathcal{O}_{\widetilde{X},E_i}((1-p^e)a_i E_i) \to \mathcal{O}_{\widetilde{X},E_i}$. In other words, if we pay close attention to our embedding into the fraction field, the divisor associated to $\widetilde{\phi}$ corresponds to $\sum -a_i E_i$ (at least for those E_i with non-positive a_i). As we've previously alluded to, one can work with anti-effective divisors too, in that case $\widetilde{\phi}$ corresponds to $-\sum a_i E_i$.

Remark 1.3. In fact, for any effective divisor E on $\widetilde{X}$, $\pi_* \mathcal{O}_{\widetilde{X}}(\lceil \sum a_i E_i \rceil + E)$ is also stabilized by ϕ.

Remark 1.4. This immediately implies the inclusion $\tau_b(R, \Delta_\phi) \subseteq \mathcal{J}(R, \Delta_\phi)$ assuming the existence of resolutions of singularities in characteristic $p > 0$. In fact, a slight modification of this implies that $\tau_b(R, \Delta, \mathfrak{a}^t) \subseteq \mathcal{J}(R, \Delta, \mathfrak{a}^t)$ under the assumption that $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein. To see this, assume that R is local notice that for every $\psi \in M^e_{\Delta, \mathfrak{a}^t}$, we have that $\Delta_\psi = \Delta_{\psi'} + \frac{1}{p^e - 1} \operatorname{div}(f)$ where $\Delta_{\psi'} \geq \Delta$ and $f \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$. It easily follows from the method of the proof and Remark 1.3 above that $\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG \rceil)$ is ψ-stable.

We'd now like to relate F-pure and log canonical singularities.

Theorem 1.5. [HW02] *Suppose that $(X, \Delta, \mathfrak{a}^t)$ has F-pure singularities and that $K_X + \Delta$ is $\mathbb{Q}$-Gorenstein. Further suppose that $\pi : \widetilde{X} \to X$ is a proper birational map with $\widetilde{X}$ normal and $\mathfrak{a} \cdot \mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-G)$. Then if we write*

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tG = \sum a_i E_i$$

we have that each $a_i \geq -1$.

Proof. Without loss of generality, we may assume that X is the spectrum of a local ring. We choose $\psi \in M^e_{\Delta, \mathfrak{a}^t}$ which induces a surjective map $\psi : F^e_* R \to R$. We notice that if we write

$$K_{\widetilde{X}} - \pi^*(K_X + \Delta_\psi) = \sum b_i E_i$$

then all of the $b_i \leq a_i$ and so it suffices to prove the statement for the b_i.

Suppose then that one of the $b_i < -1$. Localize at the generic point of the associated E_i. This gives us a DVR $\mathcal{O}_{\widetilde{X}, E_i}$ and a map $\widetilde{\psi} : F^e_* \mathcal{O}_{\widetilde{X}, E_i} \to \mathcal{O}_{\widetilde{X}, E_i}$ that is also surjective. Furthermore, the divisor corresponding to $\widetilde{\psi}$ is $-b_i E_i$. Therefore, our result follows from the following lemma:

Lemma 1.6. *If (S, Δ) is F-pure with Δ effective, then $\lceil \Delta \rceil$ is reduced (in other words, the coefficients of Δ are less than or equal to 1).*

Proof. Without loss of generality we may assume that S is a DVR with parameter s. Write $\Delta = \lambda \operatorname{div}(s)$. Suppose that $\lambda > 1$, we will show that (S, Δ) is not F-pure. Let Ψ_S be the generating map of $\operatorname{Hom}_S(F^e_* S, S)$. Then for any $\phi \in M^e_\Delta$, we have $\phi(\_\_) = \Phi_S(x \cdot \_\_)$ where $x = us^m$ and $m \geq \lceil (p^e - 1)\lambda \rceil \geq p^e$. But then clearly $\phi(z) \subseteq (s)$ for all $z \in F^e_* S$ proving that no ϕ can be surjective. $\square$

$\square$

Corollary 1.7. [MvdK92] *Suppose that X is a normal variety and $\pi : \widetilde{X} \to X$ is a projective birational map with normal $\widetilde{X}$. If there exists a map $\phi : F^e_* \mathcal{O}_X \to \mathcal{O}_X$ such that*

(a) *$(X, \Delta_\phi) = (X, \phi)$ is strongly F-regular.*
(b) *If we write $K_{\widetilde{X}} - \pi^*(K_X + \Delta) = \sum a_i E_i$ then all a_i satisfy $-1 < a_i \leq 0$ (note the lower bound follows from (a)).*

Then $R^i \pi_* \omega_{\widetilde{X}} = 0$ for all $i > 0$. In fact, $R^i \pi_* h^j(\omega^{\bullet}_{\widetilde{X}}) = 0$ for all j.

Proof. The statement is local so we may assume that X is the spectrum of a local ring R. Fix an anti-effective relatively π-ample Weil divisor E on $\widetilde{X}$ and choose an element $d \in R$ such that $\operatorname{div}_{\widetilde{X}}(d) \geq -E$. By the first hypothesis, there exists an $n \gg 0$ such that $\phi^n(F^{ne}_* dR) = R$ say $\phi^n(F^{ne}_* dc) = 1$. Consider the map $\psi : F^{ne}_* R \to R$ defined by $\phi(\_\_) = \phi(cd \cdot \_\_)$, noting that $\Delta_\psi \geq \Delta_\phi$. Write $K_{\widetilde{X}} - \pi^*(K_X + \Delta) = \sum b_i E_i$ and observe that $-1 \leq b_i < 0$ (actually, $b_i = a_i - \frac{1}{p^{ne}-1} \operatorname{div}_{E_i}(cd)$). We also induce a map $\widetilde{\psi} : F^e_* \mathcal{O}_{\widetilde{X}}((1-p^{ne}) \sum b_i E_i) \to \mathcal{O}_{\widetilde{X}}$ which sends 1 to 1. All of the a_i and b_i are non-positive, and so we have an inclusion $\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}((1-p^{ne}) \sum b_i E_i)$. In fact, by construction we have that

$$\mathcal{O}_{\widetilde{X}} \subseteq \mathcal{O}_{\widetilde{X}}(-E) \subseteq \mathcal{O}_{\widetilde{X}}(\operatorname{div}_{\widetilde{X}}(d)) \subseteq \mathcal{O}_{\widetilde{X}}((1-p^{ne}) \sum b_i E_i).$$

In particular, $\mathcal{O}_{\widetilde{X}}$ is Frobenius split, and we can express the splitting as the isomorphism

$$\mathcal{O}_{\widetilde{X}} \to F^{ne}_* \mathcal{O}_{\widetilde{X}} \to F^{ne}_* \mathcal{O}_{\widetilde{X}}(-E) \to \mathcal{O}_{\widetilde{X}}.$$

Iterating this isomorphism m-times, we obtain the isomorphism

$$\mathcal{O}_{\widetilde{X}} \to F^{mne}_* \mathcal{O}_{\widetilde{X}} \to F^{mne}_* \mathcal{O}_{\widetilde{X}}(-(1+p+\cdots+p^{m-1})E) \to \mathcal{O}_{\widetilde{X}}$$

The idea will be we can use Frobenius to amplify the amplitude of E.

Dualizing, we obtain that

$$\omega^{\bullet}_{\widetilde{X}} \leftarrow F^{mne}_* \omega^{\bullet}_{\widetilde{X}} \leftarrow F^{mne}_* \omega^{\bullet}_{\widetilde{X}}((1+p+\cdots+p^{m-1})E) \leftarrow \omega^{\bullet}_{\widetilde{X}}$$

also an isomorphism. Taking cohomology gives us an isomorphism

$$h^j(\omega^{\bullet}_{\widetilde{X}}) \leftarrow F^{mne}_* h^j(\omega^{\bullet}_{\widetilde{X}}) \leftarrow F^{mne}_* h^j(\omega^{\bullet}_{\widetilde{X}})((1+p+\cdots+p^{m-1})E) \leftarrow h^j(\omega^{\bullet}_{\widetilde{X}}).$$

Applying $R^i\pi_*$ gives us the desired conclusion since E is anti-ample and we may take $m \gg 0$. ☐

We now relate the multiplier ideal and the test ideal.

Theorem 1.8. [Smi00], [Har05], [HY03], [Tak04] *Suppose that* $(X_0 = \operatorname{Spec} R_0, \Delta_0, \mathfrak{a}_0^t)$ *is a triple in characteristic zero such that* $K_{X_0} + \Delta_0$ *is* $\mathbb{Q}$*-Cartier. Then for all* $p \gg 0$, $(\mathcal{J}(X, \Delta, \mathfrak{a}^t))_p = \tau(X_p, \Delta_p, \mathfrak{a}_p^t)$.

Proof. We will be doing reduction to characteristic $p > 0$ here. We will not write the subscript p (although will write the subscript 0). We first recall Hara's lemma on surjectivity of the dual Frobenius map (which we still haven't proved).

Lemma 1.9. [Har98] *Suppose that* R_0 *is a ring of characteristic zero,* $\pi : \widetilde{X}_0 \to \operatorname{Spec} R_0$ *is a log resolution of singularities,* D_0 *is a* π*-ample* $\mathbb{Q}$*-divisor with simple normal crossings support. We reduce this setup to characteristic* $p \gg 0$*. Then the natural map*

$$(F^e)^\vee = \Phi_{\widetilde{X}} : F^e_* \omega_{\widetilde{X}}(\lceil p^e D \rceil) \to \omega_{\widetilde{X}_p}(\lceil D \rceil)$$

surjects.

Fixing a log resolution $\widetilde{X}_0$ of X_0 we write $\mathfrak{a}_0 \cdot \mathcal{O}_{\widetilde{X}_0} = \mathcal{O}_{\widetilde{X}_0}(-G_0)$ and reduce this setup to characteristic $p > 0$. We choose $c_0 \in \mathcal{O}_{X_0}$ an element whose power is going to be a test element in characteristic $p \gg 0$, and then further multiply it by the product of the generators of the a_i. We choose a relatively ample divisor exceptional E_0 in characteristic zero such that $\lceil -\pi^*(K_{X_0} + \Delta_0) - tG_0 + E_0 - \varepsilon \operatorname{div}_{\widetilde{X}_0}(c_0) \rceil = \lceil -\pi^*(K_{X_0} + \Delta_0) - tG_0 + E_0 \rceil$ and also reduce it to characteristic $p > 0$. Our D_0 is going to be $E_0 - \pi^(K_{X_0} + \Delta_0) - tG_0 - \varepsilon \operatorname{div}_{\widetilde{X}_0}(c_0)$.

After reduction to characteristic $p \gg 0$, we may assume that $K_X + \Delta_X$ is $\mathbb{Q}$-Cartier with index not divisible by p. Therefore, we may choose a $\phi : F^e_* R \to R$ corresponding to Δ_X as before. As we've noted, this induces a map

$$\widetilde{\phi} : F^e_* \omega_{\widetilde{X}}(\lceil -\pi^*(K_X+\Delta) - tp^e G + p^e E + p^e \varepsilon \operatorname{div}_{\widetilde{X}}(c) \rceil) \to \omega_{\widetilde{X}}(\lceil -\pi^*(K_X+\Delta) - tG + E + \varepsilon \operatorname{div}_{\widetilde{X}}(c) \rceil)$$

We claim that this map can be identified with:

$$(F^e)^\vee : F^e_*\omega_{\widetilde{X}}(\lceil -p^e\pi^*(K_X+\Delta)-tp^eG+p^eE+p^e\varepsilon\operatorname{div}_{\widetilde{X}}(c)\rceil) \to \omega_{\widetilde{X}}(\lceil -\pi^*(K_X+\Delta)-tG+E+\varepsilon\operatorname{div}_{\widetilde{X}}(c)\rceil)$$

Given this claim, $\widetilde{\phi}$ surjects. Now argue as we did for rational singularities. For $e \gg 0$, π_* of the domain of $\widetilde{\phi}$ is contained inside

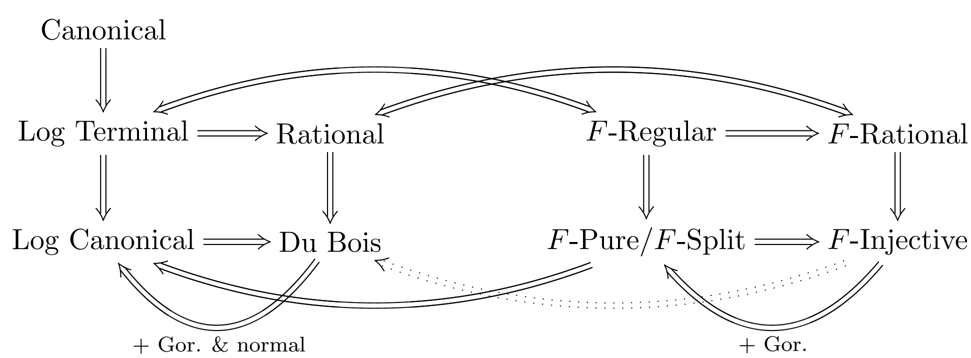
$$F^e_* c^n \overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil}}$$

where c^{n-1} is a test element. The problem is the integral closure. We need $c\overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil}} \subseteq \mathfrak{a}^{\lceil t(p^e-1)\rceil}$. But c factors as both a test element d of R as well as the product of generators of $\mathfrak{a}$. Therefore, $c\overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil}} \subseteq d\overline{\mathfrak{a}^{\lceil t(p^e-1)\rceil+r}}$ where r is the number of generators of R. The tight-closure Briancon-Skoda theorem (which we may prove a little later, []) tells us that this is contained in $\mathfrak{a}^{\lceil t(p^e-1)\rceil}$ as desired. Then the sum of images of these maps (for $e \gg 0$) is the test ideal.

To prove the claim, we argue as follows. Notice first that $(F^e)^\vee : F^e_*\mathcal{O}_{\widetilde{X}}((1-p^e)K_{\widetilde{X}}) \to \mathcal{O}_{\widetilde{X}}$ is (locally) the generating map as is $\widetilde{\phi} : F^e_*\mathcal{O}_{\widetilde{X}}((p^e-1)\pi^*(K_X+\Delta)-(p^e-1)K_{\widetilde{X}}) \to \mathcal{O}_{\widetilde{X}}$. But $\mathcal{O}_{\widetilde{X}}(K_{\widetilde{X}}+(p^e-1)\pi^*(K_X+\Delta)) \cong F^e_*\mathcal{O}_{\widetilde{X}}((1-p^e)K_{\widetilde{X}})$ so the two maps are actually the same (up to multiplication by a unit). From there, the more complicated maps above were then obtained by twisting by the same $\mathbb{Q}$-divisors, and then doing the same inclusions. ☐

Corollary 1.10. *A triple $(X, \Delta, \mathfrak{a}^t)$ in characteristic zero is Kawamata log terminal if and only if it is of open strongly F-regular type.*

Remark 1.11. The following diagram explains the singularities we understand and the implications between them.



It is an open question whether Du Bois singularities have dense F-injective type or whether log canonical singularities have dense F-pure type.

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