F-SINGULARITIES AND FROBENIUS SPLITTING NOTES EXERCISE SET #1

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(1) Prove that R is F-split if and only if it is F^e-split (for some $e > 0$).
(2) Let R be an F-finite ring. Show that if for every maximal ideal $\mathfrak{m} \in \operatorname{Spec} R$, $R_{\mathfrak{m}}$ is F-split, then R is also F-split.
Hint: Consider the evaluation-at-1 map $\operatorname{Hom}_R(F_* R, R) \to R$.
(3) Suppose S is an F-finite regular ring. If $R = S/I$ is Frobenius split, show that S is also compatibly Frobenius split with I.
Hint: First do the case when S is local, then proceed as in (1).
(4) Are the following rings semi-normal and/or weakly normal? Are they F-split?
  (a) $\mathbb{F}_p(x^p)[y, xy, x^2y, \dots, x^{p-1}y]$
  (b) $\mathbb{F}_p[u, v, y, z]/(uv, uz, z(v - y^2))$ (*Hint:* It might help to study what the irreducible components of the Spec of this ring are).
  (c) Any union of coordinate linear spaces through the origin in $\mathbb{A}^n$.
  (d) $\mathbb{F}_p[x, y, z, w]/(x^3 + y^3 + z^3 + w^3)$ (check this for $p = 2, 3, 5, 7, 11$).
(5) If X is F-split, show that every irreducible component of X is also F-split.
(6) Suppose that $R = S/I$ is a Gorenstein ring and S is an F-finite regular local ring. Show that $I^{[p^e]} : I = I^{[p^e]} + (f)$ for some element $f \in S$. More generally, show that the same conclusion holds if R is normal and $(p^e - 1)K_{\operatorname{Spec} R}$ is Cartier.
(7) Suppose that $R = S/I$ is a complete intersection where S is an F-finite regular local ring (in other words, $I = (x_1, \dots, x_n)$ is generated by a regular sequence). Show that $I^{[p^e]} : I = I^{[p^e]} + (x_1^{p^e-1} \cdot \dots \cdot x_n^{p^e-1})$. State an easy to check criteria for R to be F-split.
(8) If $K \subseteq L$ is a finite separable extension of F-finite fields, show that every K-linear map $\phi : F^e_* K \to K$ extends to an L-linear map $\bar{\phi} : F^e_* L \to L$.
Hint: Show first that any basis for $F^e_* K$ over K is also a basis for $F^e_* L$ over L.
(9) Suppose that $S = k[x_1, \dots, x_n]$ and that $\Phi : F_* S \to S$ is the generating map (as discussed in class, it sends $x_1^{p-1} \dots x_n^{p-1}$ to 1 and the other monomials to zero). Suppose that $\phi : F_* S \to S$ is any other S-linear map with $\phi(\_\_) = \Phi(z \cdot \_\_)$ for some $z \in F_* S$. Show that ϕ is compatible with $R = S/(f)$ if and only if f^{p-1} divides z.