Characteristic 0 closure operations similar to tight closure

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JMM 2026, Washington D.C.

Tight closure?

Definition: [Hochster-Huneke]

Suppose R is a Noetherian F-finite domain of char p, $I \subseteq R$ ideal

$$I^* = \{x \in R \mid \exists c \neq 0, c^{1/p^e} x \in IR_{\text{perf}}, e \gg 0\}.$$

This is the *tight closure of I.*

It is the elements *almost* in

$$IR_{\text{perf}} = I\left(\bigcup_e R^{1/p^e}\right)$$

For some $0 \neq c \in R$

Many good properties…

Properties of tight closure

- Closure, pres. containments & $I \subseteq I^* = (I^*)^*$
- Colon capturing $f_1, \dots f_n$ s.o.p $\Rightarrow$

$$(f_1, \dots, f_i)^* : f_{i+1} \subseteq (f_1, \dots, f_i)^*.$$

- Detects singularities

  $I^* = I$ for parameter ideals

  is close to *rational singularities*
- Finite maps $R \subseteq S$ finite then

  $I^* = (IS)^* \cap R$
- Briançon-Skoda Theorem

  $I = (f_1, \dots, f_n)$ then

$$\overline{I^{n+k-1}} \subseteq (I^k)^*$$

- But challenging to compute, doesn't commute with localization (Brenner-Monsky).

What about other characteristics?

- In characteristic *p*, plus closure has many same properties. $I^+ = IR^+ \cap R$
- In mixed characteristic, use big Cohen-Macaulay algebra extension-contraction (char free) or Heitmann's epf closure.
- In characteristic zero, do reduction mod *p* or Brenner's parasolid closure (char free).
- In characteristic zero, tight closure implies theorems also obtained via resolution of sings and Kodaira vanishing (dictionary).
- **Is there a resolution of sings closure?**

The idea

Instead of expanding contracting from R^+, R_{perf} parasolid algebras, or big Cohen-Macaulay algebras we should extend and contract from

$$\mathbb{R}\Gamma(Y, \mathcal{O}_Y)$$

for $Y \to \operatorname{Spec} R$ a resolution of singularities.

$$\mathbb{R}\Gamma(Y, \mathcal{O}_Y) \text{ is}$$

- **Independent of the choice of resolution**
- **A differential graded (cosimplicial) *R*-algebra**
- **A Cohen-Macaulay complex**

How to expand and contract from a DG algebra?

OPTION #1

- *Hironaka closure (Hironaka proved res. of sings)*

$$
\begin{aligned}
I^{\mathrm{Hir}} &= \ker\Big(R \to H_0\big(R/I \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_Y)\big)\Big) \\
&= \mathrm{Ann}\big(R/I \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_Y)\big)
\end{aligned}
$$

OPTION #2

- *Koszul-Hironaka closure*

$$I = (f_1, \dots, f_n) = (\underline{f})$$

$$
\begin{aligned}
I^{\mathrm{KH}} &= \ker\Big(R \to H_0\big(\mathrm{Kos}(\underline{f}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_Y)\big)\Big) \\
&= \mathrm{Ann}\big(\mathrm{Kos}(\underline{f}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_Y)\big)
\end{aligned}
$$

Independent of choices.

	Tight closure	Hironaka	Koszul-Hironaka
Closure	YES	?	YES
Colon Capturing	YES	YES	YES
Detects singularities	YES	YES	YES
Finite Maps	YES	?	YES
Briançon-Skoda	YES	YES (M.-M.-R.G.-S.)	Weak version
Commutes localization?	NO	YES	YES
Computable?	Hard	For CM rings	YES

By computable, we really mean a computer can compute it. The Macaulay2 package associated with the paper really lets you experiment.

Compare the operations

Fix R finite type over a field of char 0. Let I^* denote elements in tight closure after reduction mod $p \gg 0$.

Theorem: E.-M.-R.G.-S.

$$I^{\mathrm{KH}} \subseteq I^{\mathrm{Hir}} \subseteq I^*.$$

Furthermore, all agree if I generated by s.o.p.

In general, containments can be strict. We have one more operation not defined here bigger than I^*.

Briançon-Skoda versions

Theorem: E.-M.-R.G.-S.

$I = (f_1, \ldots, f_n)$ then

$$\overline{I^n} \subseteq I^{\mathrm{KH}}$$

But $I^{n+1} \not\subseteq (I^2)^{\mathrm{KH}}$ (via computer)

Theorem: M.-M.-R.G.-S.

$I = (f_1, \ldots, f_n)$ then $\overline{I^{n+k-1}}$ maps to zero in

$$H_0\Big(L^k(\underline{f}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_Y)\Big)$$

Where $L^k(\underline{f})$ is Buchsbaum-Eisenbud or Eagon-Northcott complex. Hence:

$$\overline{I^{n+k-1}} \subseteq (I^k)^{\mathrm{Hir}}.$$

Surpise!

Those versions hold in a derived, char. free, non-Noetherian environment & more!

Theorem: M.-M.-R.G.-S.

$I = (f_1, \ldots, f_n)$ let $X \to \mathrm{Spec} R$ be blowup of $\overline{I^{n+k-1}}$ with exceptional E. Then

$$\mathbb{R}\Gamma(\mathcal{O}_X(-E)) \to L^k(\underline{f}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_X)$$

is zero in derived category. Where $L^k(\underline{f})$ is Buchsbaum-Eisenbud or Eagon-Northcott complex[a].

Taking 0th homology, $\overline{I^{n+k-1}} \mapsto 0$ in

$$H_0\Big(L^k(\underline{f}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_X)\Big)$$

[a] agrees with Koszul if $k = 1$

Expanation: Instead of blowup of *I*

If $Y \to \mathrm{Spec} R$ is a:

- resolution of singularities, or
- regular alteration, or
- pseudo-rational alteration, or
- regular alteration hypercover:

Then there are natural

$$R \to L^k(\underline{f}) \to R/I^k,$$

$$R \to \mathbb{R}\Gamma(\mathcal{O}_X) \to \mathbb{R}\Gamma(\mathcal{O}_Y).$$

Get closure Briançon-Skoda indep. of I (any char)

$$\overline{I^{n+k-1}} \mapsto 0 \in H_0(L^k(\underline{f}) \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_X)) \to H_0(R/I^k \otimes^{\mathbb{L}} \mathbb{R}\Gamma(\mathcal{O}_Y)).$$

Some corollaries!

Theorem: M.-M.-R.G.-S., cf L.-T.

If R is a derived birational splinter (for instance pseudo-rational), then

$$\overline{I^{n+k-1}} \subseteq I^k.$$

Theorem: M.-M.-R.G.-S., cf H.-W.

If R is a normal exc. blowup-square splinter (ie Du Bois, F-pure, perfectoid pure), then

$$\overline{I_{>n+k-1}} \subseteq I^k.$$

- Also, recovers tight closure, plus closure, epf closure, versions of Briançon-Skoda

More Corollaries

Theorem: M.-M.-R.G.-S., cf Huneke

If R is reduced, finite dim. quasi-exc., then there exists d such that $\forall k \geq 1$, $\forall I$,

$$\overline{I^{d+k}} \subseteq I^k.$$

- Holds since regular alteration hypercover is good enough, and those exist by Gabber.
- We also get uniform Artin-Rees (we proved missing piece, then used Huneke).

Theorem: M.-M.-R.G.-S., cf Huneke

If R is finite dim. quasi-exc. $N \subseteq M$ f.g., $\exists \ell$ such that $\forall n \geq \ell$, $\forall I$,

$$I^n M \cap N \subseteq I^{n-\ell} M.$$

Thank you for listening!

Questions?