Singularities in characteristic zero and singularities in characteristic p

Karl Schwede[1]

[1] Department of Mathematics
University of Michigan

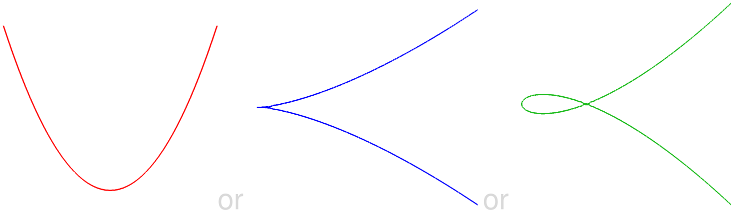
Special Lecture

Outline

Outline

Affine algebraic varieties

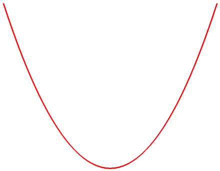
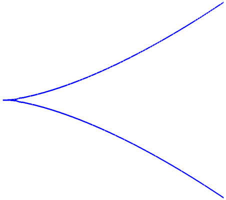
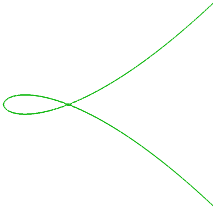
- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x-1)$.



- Of course, these are two dimensional objects really, we only plotted their real points.

Affine algebraic varieties

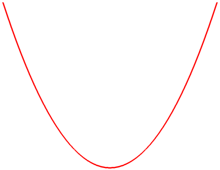
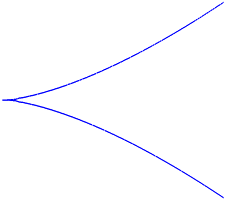
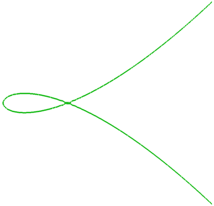
- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x-1)$.


or

or


- Of course, these are two dimensional objects really, we only plotted their real points.

Affine algebraic varieties

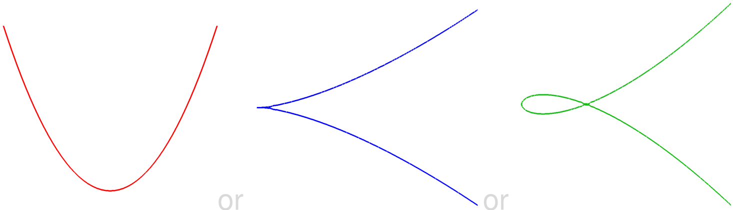
- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x-1)$.

 or  or 

- Of course, these are two dimensional objects really, we only plotted their real points.

Affine algebraic varieties

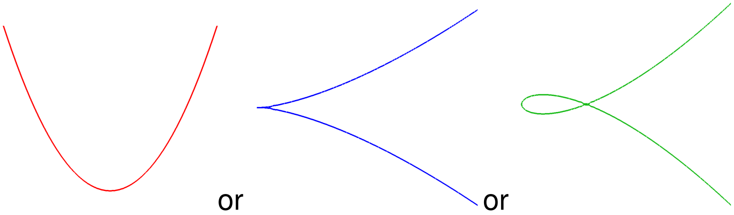
- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x-1)$.



- Of course, these are two dimensional objects really, we only plotted their real points.

Affine algebraic varieties

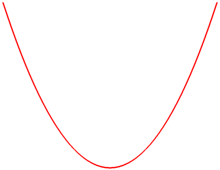
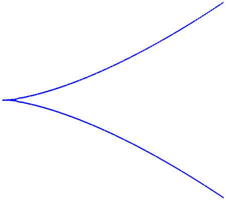
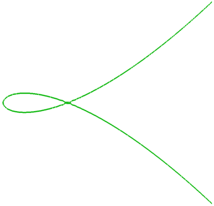
- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x - 1)$.



- Of course, these are two dimensional objects really, we only plotted their real points.

Affine algebraic varieties

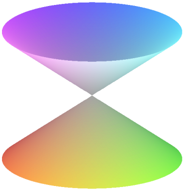
- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x - 1)$.

 or  or 

- Of course, these are two dimensional objects really, we only plotted their real points.

Higher dimensional examples I

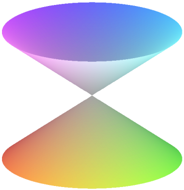
- In $\mathbb{C}^3$ one might consider a quadric cone, $x^2 + y^2 - z^2$.



- Or a cone over a cubic, $y^2 z - x(x - z)(x + z)$.

Higher dimensional examples I

- In $\mathbb{C}^3$ one might consider a quadric cone, $x^2 + y^2 - z^2$.



- Or a cone over a cubic, $y^2 z - x(x - z)(x + z)$.

Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_{\mathbb{C}}$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.
  - There won't be any positive characteristic drawings.

Generalizations

- These examples are not compact (they are affine). Often one studies "projective" algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_{\mathbb{C}}$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.
  - There won't be any positive characteristic drawings.

Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_{\mathbb{C}}$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.
  - There won't be any positive characteristic drawings.

Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_{\mathbb{C}}$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.
  - There won’t be any positive characteristic drawings.

Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
    - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_{\mathbb{C}}$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.
    - There won't be any positive characteristic drawings.

Relation with algebra

- If one is studying a complex affine variety X defined by an equation $f(x_1, \ldots, x_n) = 0$, the ring

$$R = \mathbb{C}[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$$

carries the same information as X (although it doesn't record the embedding $X \subseteq \mathbb{C}^n$).

- The points of the variety correspond to the maximal ideals of the ring R.
- Therefore, one can study the algebraic variety X by studying the ring R.
- This is particularly useful when working over fields besides $\mathbb{C}$.

Relation with algebra

- If one is studying a complex affine variety X defined by an equation $f(x_1, \ldots, x_n) = 0$, the ring

$$R = \mathbb{C}[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$$

carries the same information as X (although it doesn't record the embedding $X \subseteq \mathbb{C}^n$).

- The points of the variety correspond to the maximal ideals of the ring R.
- Therefore, one can study the algebraic variety X by studying the ring R.
- This is particularly useful when working over fields besides $\mathbb{C}$.

Relation with algebra

- If one is studying a complex affine variety X defined by an equation $f(x_1, \dots, x_n) = 0$, the ring

$$R = \mathbb{C}[x_1, \dots, x_n]/(f(x_1, \dots, x_n))$$

carries the same information as X (although it doesn't record the embedding $X \subseteq \mathbb{C}^n$).
- The points of the variety correspond to the maximal ideals of the ring R.
- Therefore, one can study the algebraic variety X by studying the ring R.
- This is particularly useful when working over fields besides $\mathbb{C}$.

Relation with algebra

- If one is studying a complex affine variety X defined by an equation $f(x_1, \dots, x_n) = 0$, the ring

$$R = \mathbb{C}[x_1, \dots, x_n]/(f(x_1, \dots, x_n))$$

carries the same information as X (although it doesn't record the embedding $X \subseteq \mathbb{C}^n$).
- The points of the variety correspond to the maximal ideals of the ring R.
- Therefore, one can study the algebraic variety X by studying the ring R.
- This is particularly useful when working over fields besides $\mathbb{C}$.

This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point Q is *smooth* if "very locally", that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if X is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point Q is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we've looked at so far (except the parabola) have an "isolated singularity" at the origin.

This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point Q is *smooth* if “very locally”, that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if X is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point Q is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.

This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point Q is *smooth* if "very locally", that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if X is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point Q is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we've looked at so far (except the parabola) have an "isolated singularity" at the origin.

This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point Q is *smooth* if "very locally", that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if X is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point Q is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we've looked at so far (except the parabola) have an "isolated singularity" at the origin.

This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point Q is *smooth* if "very locally", that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if X is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point Q is singular if $f(Q) = 0$ and $\partial f/\partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we've looked at so far (except the parabola) have an "isolated singularity" at the origin.

This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point Q is *smooth* if "very locally", that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if X is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point Q is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we've looked at so far (except the parabola) have an "isolated singularity" at the origin.

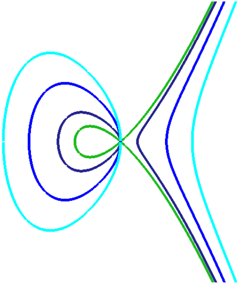
This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point Q is *smooth* if "very locally", that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if X is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point Q is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we've looked at so far (except the parabola) have an "isolated singularity" at the origin.

Why study singularities? I

Perhaps you are only interested in smooth varieties?

- Singularities show up as limits of smooth varieties.

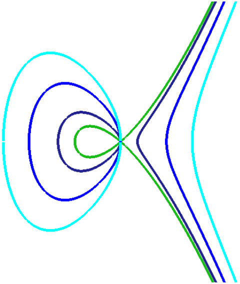


- This happens particularly when “compactifying moduli spaces”
  - (moduli spaces are algebraic varieties whose points parameterize something. For example, points can correspond to isomorphism classes of certain varieties).

Why study singularities? I

Perhaps you are only interested in smooth varieties?

- Singularities show up as limits of smooth varieties.

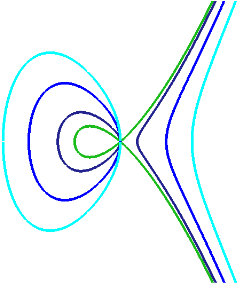


- This happens particularly when “compactifying moduli spaces”
  - (moduli spaces are algebraic varieties whose points parameterize something. For example, points can correspond to isomorphism classes of certain varieties).

Why study singularities? I

Perhaps you are only interested in smooth varieties?

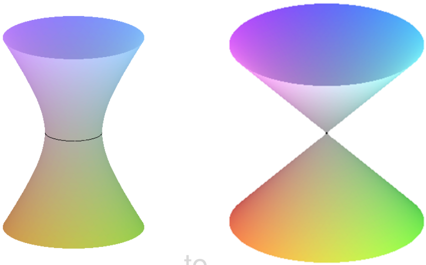
- Singularities show up as limits of smooth varieties.



- This happens particularly when “compactifying moduli spaces”
  - (moduli spaces are algebraic varieties whose points parameterize something. For example, points can correspond to isomorphism classes of certain varieties).

Why study singularities? II

- If you want to classify algebraic varieties, sometimes you need to replace a variety X with a simpler but closely related variety Y.
- One way in which this is done is by contracting (compact) subsets of varieties to points.

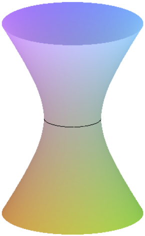


to

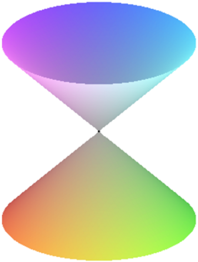
- This happens in the minimal model program.

Why study singularities? II

- If you want to classify algebraic varieties, sometimes you need to replace a variety X with a simpler but closely related variety Y.
- One way in which this is done is by contracting (compact) subsets of varieties to points.



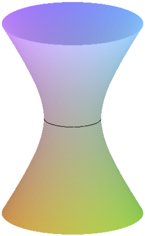
to



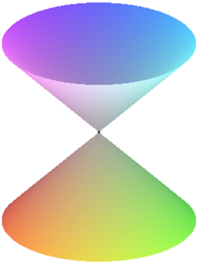
- This happens in the minimal model program.

Why study singularities? II

- If you want to classify algebraic varieties, sometimes you need to replace a variety X with a simpler but closely related variety Y.
- One way in which this is done is by contracting (compact) subsets of varieties to points.



to



- This happens in the minimal model program.

Why study singularities? III

- Of course, sometimes you simply want to generalize a theorem to as broad a setting as possible, and so you ask
- "What property of smooth varieties allows me to prove this theorem?"
- Once you can answer this question, you have identified a class of singularities.

Why study singularities? III

- Of course, sometimes you simply want to generalize a theorem to as broad a setting as possible, and so you ask
- “What property of smooth varieties allows me to prove this theorem?”
- Once you can answer this question, you have identified a class of singularities.

Why study singularities? III

- Of course, sometimes you simply want to generalize a theorem to as broad a setting as possible, and so you ask
- “What property of smooth varieties allows me to prove this theorem?”
- Once you can answer this question, you have identified a class of singularities.

Outline

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)

    Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

What is a resolution of singularities?

- Suppose you are given a singular variety X.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \widetilde{X} \to X$ that satisfies the following properties:
  - $\widetilde{X}$ is smooth.
  - π is "birational" (this means it is an isomorphism outside of a small closed subset of X, usually the singular locus of X)
  - π is "proper" (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\widetilde{X}$ is usually not affine, even when X is.
  - We also usually require that the pre-image of the singular locus looks like "coordinate hyperplanes", sufficiently locally.
- Resolutions of singularities always exist in characteristic zero

Why resolve singularities?

- A resolution of singularities takes your variety X and constructs a "smooth variety" $\widetilde{X}$ that is very closely related to X.
  - $\widetilde{X}$ and X are "birational".
- The "properness" of the resolution implies that if X was compact, then $\widetilde{X}$ is also compact.
- So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.

Why resolve singularities?

- A resolution of singularities takes your variety X and constructs a "smooth variety" $\widetilde{X}$ that is very closely related to X.
  - $\widetilde{X}$ and X are "birational".
- The "properness" of the resolution implies that if X was compact, then $\widetilde{X}$ is also compact.
- So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.

Why resolve singularities?

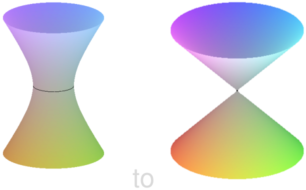
- A resolution of singularities takes your variety X and constructs a "smooth variety" $\widetilde{X}$ that is very closely related to X.
  - $\widetilde{X}$ and X are "birational".
- The "properness" of the resolution implies that if X was compact, then $\widetilde{X}$ is also compact.
- So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.

Why resolve singularities?

- A resolution of singularities takes your variety X and constructs a “smooth variety” $\widetilde{X}$ that is very closely related to X.
  - $\widetilde{X}$ and X are “birational”.
- The “properness” of the resolution implies that if X was compact, then $\widetilde{X}$ is also compact.
- So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.

How do you resolve singularities?

- You perform several *blow-ups*.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

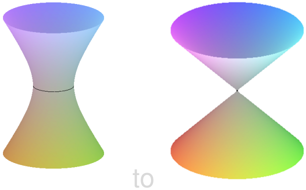


Theorem (Hironaka)

In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities

How do you resolve singularities?

- You perform several *blow-ups*.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

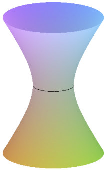
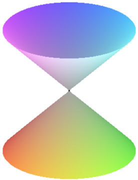


Theorem (Hironaka)

In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities

How do you resolve singularities?

- You perform several *blow-ups*.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

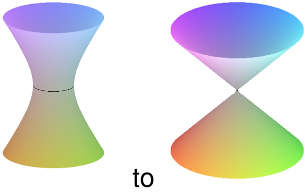

to


Theorem (Hironaka)

In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities

How do you resolve singularities?

- You perform several *blow-ups*.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

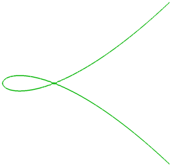


Theorem (Hironaka)

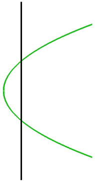
In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities

Example with curves

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at Q into its own point. It replaces Q by a copy of $\mathbb{P}^1_{\mathbb{C}}$ = "The Riemann sphere".
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.



blown-up becomes



- The black line is the $\mathbb{P}^1_{\mathbb{C}}$ that will be contracted back to the origin in $\mathbb{C}^2$.

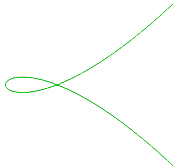
Example with curves

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at Q into its own point. It replaces Q by a copy of $\mathbb{P}^1_{\mathbb{C}}$ = "The Riemann sphere".
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.

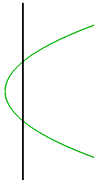
- The black line is the $\mathbb{P}^1_{\mathbb{C}}$ that will be contracted back to the origin in $\mathbb{C}^2$.

Example with curves

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at Q into its own point. It replaces Q by a copy of $\mathbb{P}^1_{\mathbb{C}}$ = "The Riemann sphere".
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.



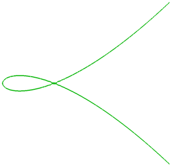
blown-up becomes



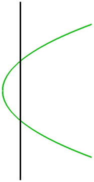
- The black line is the $\mathbb{P}^1_{\mathbb{C}}$ that will be contracted back to the origin in $\mathbb{C}^2$.

Example with curves

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at Q into its own point. It replaces Q by a copy of $\mathbb{P}^1_{\mathbb{C}} =$ "The Riemann sphere".
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.



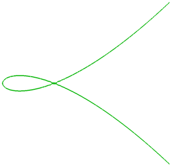
blown-up becomes



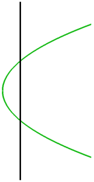
- The black line is the $\mathbb{P}^1_{\mathbb{C}}$ that will be contracted back to the origin in $\mathbb{C}^2$.

Example with curves

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at Q into its own point. It replaces Q by a copy of $\mathbb{P}^1_{\mathbb{C}}$ = "The Riemann sphere".
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.



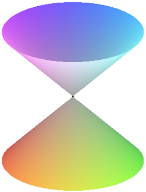
blown-up becomes



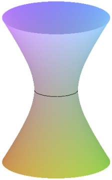
- The black line is the $\mathbb{P}^1_{\mathbb{C}}$ that will be contracted back to the origin in $\mathbb{C}^2$.

Additional discussion of blow-ups

- A similar thing happens with the quadric cones in $\mathbb{C}^3$.



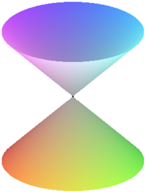
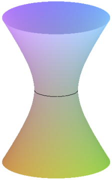
to



- When we do the blow-up at the origin, all the different tangent directions get separated.
- But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle.
  - at least its real points look like a circle.

Additional discussion of blow-ups

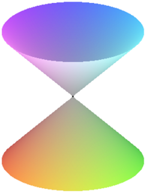
- A similar thing happens with the quadric cones in $\mathbb{C}^3$.

 to 

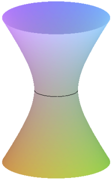
- When we do the blow-up at the origin, all the different tangent directions get separated.
- But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle.
  - at least its real points look like a circle.

Additional discussion of blow-ups

- A similar thing happens with the quadric cones in $\mathbb{C}^3$.



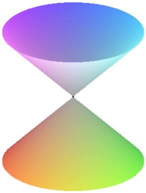
to



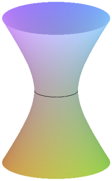
- When we do the blow-up at the origin, all the different tangent directions get separated.
- But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle.
  - at least its real points look like a circle.

Additional discussion of blow-ups

- A similar thing happens with the quadric cones in $\mathbb{C}^3$.



to



- When we do the blow-up at the origin, all the different tangent directions get separated.
- But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle.
  - at least its real points look like a circle.

How can we classify singularities with resolutions?

- All the examples we've seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.
- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
    - You can do something like this for surfaces (surface = 2 complex dimensions).
- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different "minimal" ways to resolve the same singularity.
- You can often compare the (geometric / algebraic / homological) properties of the resolution $\widetilde{X}$ with those same (geometric / algebraic / homological) properties of X.

How can we classify singularities with resolutions?

- All the examples we've seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.
- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).
- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different "minimal" ways to resolve the same singularity.
- You can often compare the (geometric / algebraic / homological) properties of the resolution $\widetilde{X}$ with those same (geometric / algebraic / homological) properties of X.

How can we classify singularities with resolutions?

- All the examples we've seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.
- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).
- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different "minimal" ways to resolve the same singularity.
- You can often compare the (geometric / algebraic / homological) properties of the resolution $\widetilde{X}$ with those same (geometric / algebraic / homological) properties of X.

How can we classify singularities with resolutions?

- All the examples we've seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.
- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).
- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different "minimal" ways to resolve the same singularity.
- You can often compare the (geometric / algebraic / homological) properties of the resolution $\widetilde{X}$ with those same (geometric / algebraic / homological) properties of X.

How can we classify singularities with resolutions?

- All the examples we've seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.
- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).
- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different "minimal" ways to resolve the same singularity.
- You can often compare the (geometric / algebraic / homological) properties of the resolution $\widetilde{X}$ with those same (geometric / algebraic / homological) properties of X.

Singularities of the minimal model program I

- The goal of the minimal model program is to take a "birational equivalence class" of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with "mild" singularities.
- What does mild mean? One compares the sheaf of "top dimensional differentials" on X (naively extended over the singular locus) with the top differentials of its resolution $\widetilde{X}$.
- Singularities classified this way behave well with respect to the contractions of the minimal model program.
- Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.

Singularities of the minimal model program I

- The goal of the minimal model program is to take a "birational equivalence class" of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with "mild" singularities.
- What does mild mean? One compares the sheaf of "top dimensional differentials" on X (naively extended over the singular locus) with the top differentials of its resolution $\widetilde{X}$.
- Singularities classified this way behave well with respect to the contractions of the minimal model program.
- Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.

Singularities of the minimal model program I

- The goal of the minimal model program is to take a “birational equivalence class” of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with “mild” singularities.
- What does mild mean? One compares the sheaf of “top dimensional differentials” on X (naively extended over the singular locus) with the top differentials of its resolution $\widetilde{X}$.
- Singularities classified this way behave well with respect to the contractions of the minimal model program.
- Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.

Singularities of the minimal model program I

- The goal of the minimal model program is to take a "birational equivalence class" of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with "mild" singularities.
- What does mild mean? One compares the sheaf of "top dimensional differentials" on X (naively extended over the singular locus) with the top differentials of its resolution $\widetilde{X}$.
- Singularities classified this way behave well with respect to the contractions of the minimal model program.
- Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.

Singularities of the minimal model program II

- Recall we are defining singularities by looking at how the sheaf of top differential forms on a resolution $\widetilde{X}$ behaves compared to the sheaf of top differentials on X.
- By looking at the numerics of these comparisons, one can write down definitions of *terminal, canonical, log terminal, log canonical, rational* and *Du Bois* singularities.
- Actually, Du Bois singularities were originally defined using other methods (Hodge Theory), although we now have the following theorem.

Theorem (Kovács, –, Smith)

Suppose that X is normal and Cohen-Macaulay and $\pi : \widetilde{X} \to X$ is a (log) resolution of X with exceptional set E. Then X has Du Bois singularities if and only if $\pi_* \omega_{\widetilde{X}}(E) = \omega_X$.

Singularities of the minimal model program II

- Recall we are defining singularities by looking at how the sheaf of top differential forms on a resolution $\widetilde{X}$ behaves compared to the sheaf of top differentials on X.
- By looking at the numerics of these comparisons, one can write down definitions of *terminal, canonical, log terminal, log canonical, rational* and *Du Bois* singularities.
- Actually, Du Bois singularities were originally defined using other methods (Hodge Theory), although we now have the following theorem.

Theorem (Kovács, –, Smith)

Suppose that X is normal and Cohen-Macaulay and $\pi : \widetilde{X} \to X$ is a (log) resolution of X with exceptional set E. Then X has Du Bois singularities if and only if $\pi_* \omega_{\widetilde{X}}(E) = \omega_X$.

Singularities of the minimal model program II

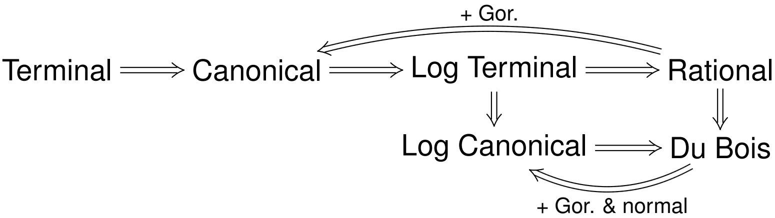
- Recall we are defining singularities by looking at how the sheaf of top differential forms on a resolution $\widetilde{X}$ behaves compared to the sheaf of top differentials on X.
- By looking at the numerics of these comparisons, one can write down definitions of *terminal, canonical, log terminal, log canonical, rational* and *Du Bois* singularities.
- Actually, Du Bois singularities were originally defined using other methods (Hodge Theory), although we now have the following theorem.

Theorem (Kovács, –, Smith)

Suppose that X is normal and Cohen-Macaulay and $\pi : \widetilde{X} \to X$ is a (log) resolution of X with exceptional set E. Then X has Du Bois singularities if and only if $\pi_* \omega_{\widetilde{X}}(E) = \omega_X$.

Singularities of the minimal model program III

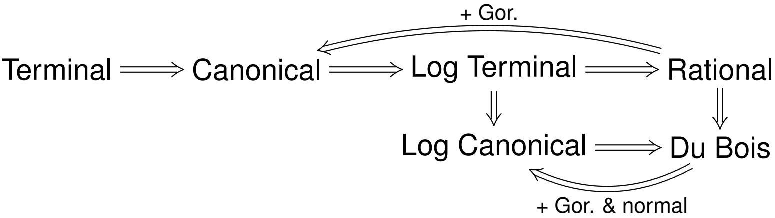
- The following diagram summarizes implications between the singularities of the minimal model program.



- Not all of the implications in the above diagram are trivial, see the work of Elkik, Ishii, Kollár, Kovács, Saito, –, Smith, Steenbrink and others.
- Multiplier ideals, adjoint ideals, log canonical thresholds and log canonical centers are also measures of singularities that fit into the same framework.

Singularities of the minimal model program III

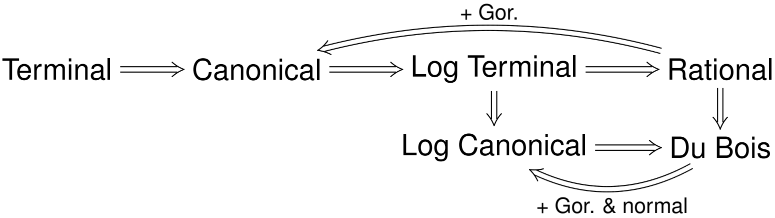
- The following diagram summarizes implications between the singularities of the minimal model program.



- Not all of the implications in the above diagram are trivial, see the work of Elkik, Ishii, Kollár, Kovács, Saito, –, Smith, Steenbrink and others.
- Multiplier ideals, adjoint ideals, log canonical thresholds and log canonical centers are also measures of singularities that fit into the same framework.

Singularities of the minimal model program III

- The following diagram summarizes implications between the singularities of the minimal model program.



- Not all of the implications in the above diagram are trivial, see the work of Elkik, Ishii, Kollár, Kovács, Saito, –, Smith, Steenbrink and others.
- Multiplier ideals, adjoint ideals, log canonical thresholds and log canonical centers are also measures of singularities that fit into the same framework.

Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.

Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.

Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.

Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.

Analytic description of singularities

- There are analytic ways to describe several of the singularities of the minimal model program as well.
- For example, consider a variety X defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.
- Also assume that f is irreducible.
- Then X is (semi) log canonical near the origin 0 if and only if

$$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}} \text{ is integrable near 0 for all } c < 1.$$

- The multiplier ideal can also be described in a similar way.

Analytic description of singularities

- There are analytic ways to describe several of the singularities of the minimal model program as well.
- For example, consider a variety X defined by an equation $f(x_1, \dots, x_n) = 0$ in $\mathbb{C}^n$.
- Also assume that f is irreducible.
- Then X is (semi) log canonical near the origin 0 if and only if

$$\frac{1}{|f(x_1, \dots, x_n)|^{2c}} \text{ is integrable near 0 for all } c < 1.$$

- The multiplier ideal can also be described in a similar way.

Analytic description of singularities

- There are analytic ways to describe several of the singularities of the minimal model program as well.
- For example, consider a variety X defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.
- Also assume that f is irreducible.
- Then X is (semi) log canonical near the origin 0 if and only if

$$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}} \text{ is integrable near 0 for all } c < 1.$$

- The multiplier ideal can also be described in a similar way.

Analytic description of singularities

- There are analytic ways to describe several of the singularities of the minimal model program as well.
- For example, consider a variety X defined by an equation $f(x_1, \dots, x_n) = 0$ in $\mathbb{C}^n$.
- Also assume that f is irreducible.
- Then X is (semi) log canonical near the origin 0 if and only if

$$\frac{1}{|f(x_1, \dots, x_n)|^{2c}} \text{ is integrable near 0 for all } c < 1.$$

- The multiplier ideal can also be described in a similar way.

Analytic description of singularities

- There are analytic ways to describe several of the singularities of the minimal model program as well.
- For example, consider a variety X defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.
- Also assume that f is irreducible.
- Then X is (semi) log canonical near the origin 0 if and only if

$$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}} \text{ is integrable near 0 for all } c < 1.$$

- The multiplier ideal can also be described in a similar way.

Outline

What's different about characteristic p?

- Suppose that k is an algebraically closed field of characteristic p.
- One can still make sense of varieties defined over k.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone's satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic p.

What's different about characteristic p?

- Suppose that k is an algebraically closed field of characteristic p.
- One can still make sense of varieties defined over k.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone's satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic p.

What's different about characteristic *p*?

- Suppose that k is an algebraically closed field of characteristic p.
- One can still make sense of varieties defined over k.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone's satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic p.

What's different about characteristic p?

- Suppose that k is an algebraically closed field of characteristic p.
- One can still make sense of varieties defined over k.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone's satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic p.

What's different about characteristic p?

- Suppose that k is an algebraically closed field of characteristic p.
- One can still make sense of varieties defined over k.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone's satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic p.

What's different about characteristic p?

- Suppose that k is an algebraically closed field of characteristic p.
- One can still make sense of varieties defined over k.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
    - Although there is hope that this might be solved to everyone's satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic p.

Study the rings

- Various people have been studying properties of rings in characteristic $p > 0$ for a long time.
- Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.
  - The Frobenius map on a ring R is the map $F : R \to R$ that sends $x \in R$ to x^p (where p is the characteristic of R).
- Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.
- If R is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.$$

Study the rings

- Various people have been studying properties of rings in characteristic $p > 0$ for a long time.
- Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.
  - The Frobenius map on a ring R is the map $F : R \to R$ that sends $x \in R$ to x^p (where p is the characteristic of R).
- Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.
- If R is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.$$

Study the rings

- Various people have been studying properties of rings in characteristic $p > 0$ for a long time.
- Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.
  - The Frobenius map on a ring R is the map $F : R \to R$ that sends $x \in R$ to x^p (where p is the characteristic of R).
- Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.
- If R is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.$$

Study the rings

- Various people have been studying properties of rings in characteristic $p > 0$ for a long time.
- Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.
  - The Frobenius map on a ring R is the map $F : R \to R$ that sends $x \in R$ to x^p (where p is the characteristic of R).
- Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.
- If R is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.$$

Study the rings

- Various people have been studying properties of rings in characteristic $p > 0$ for a long time.
- Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.
  - The Frobenius map on a ring R is the map $F : R \to R$ that sends $x \in R$ to x^p (where p is the characteristic of R).
- Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.
- If R is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.$$

Notation for Frobenius

- We want to explore the behavior of Frobenius on "nice rings"?
- We want to view R as an R-module via the action of Frobenius.
- People often use $F_* R$ to denote the R-module which is equal to R as an additive group, and where the R-module action is given by $r.x = r^p x$.
  - One can also think of $F_* R$ as $R^{1/p}$.

Notation for Frobenius

- We want to explore the behavior of Frobenius on "nice rings"?
- We want to view R as an R-module via the action of Frobenius.
- People often use $F_* R$ to denote the R-module which is equal to R as an additive group, and where the R-module action is given by $r.x = r^p x$.
  - One can also think of $F_* R$ as $R^{1/p}$.

Notation for Frobenius

- We want to explore the behavior of Frobenius on "nice rings"?
- We want to view R as an R-module via the action of Frobenius.
- People often use $F_* R$ to denote the R-module which is equal to R as an additive group, and where the R-module action is given by $r.x = r^p x$.
  - One can also think of $F_* R$ as $R^{1/p}$.

Notation for Frobenius

- We want to explore the behavior of Frobenius on "nice rings"?
- We want to view R as an R-module via the action of Frobenius.
- People often use $F_* R$ to denote the R-module which is equal to R as an additive group, and where the R-module action is given by $r.x = r^p x$.
  - One can also think of $F_* R$ as $R^{1/p}$.

Notation for Frobenius

- We want to explore the behavior of Frobenius on "nice rings"?
- We want to view R as an R-module via the action of Frobenius.
- People often use $F_* R$ to denote the R-module which is equal to R as an additive group, and where the R-module action is given by $r.x = r^p x$.
  - One can also think of $F_* R$ as $R^{1/p}$.

Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring R would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It's easy to see that $F_* R$ is free of rank p (with generators $1, x, \dots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

Theorem (Kunz)

A local domain R of characteristic p is regular (ie, non-singular) if and only if $F_* R$ is flat as an R-module.

- In our context, this implies that R is smooth if and only if $F_* R$ is locally free.

Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring R would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It's easy to see that $F_* R$ is free of rank p (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

Theorem (Kunz)

A local domain R of characteristic p is regular (ie, non-singular) if and only if $F_* R$ is flat as an R-module.

- In our context, this implies that R is smooth if and only if $F_* R$ is locally free.

Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring R would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It's easy to see that $F_* R$ is free of rank p (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

Theorem (Kunz)

A local domain R of characteristic p is regular (ie, non-singular) if and only if $F_* R$ is flat as an R-module.

  - In our context, this implies that R is smooth if and only if $F_* R$ is locally free.

Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring R would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It's easy to see that $F_* R$ is free of rank p (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

Theorem (Kunz)

A local domain R of characteristic p is regular (ie, non-singular) if and only if $F_* R$ is flat as an R-module.

  - In our context, this implies that R is smooth if and only if $F_* R$ is locally free.

Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring R would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It's easy to see that $F_* R$ is free of rank p (with generators $1, x, \dots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

Theorem (Kunz)

A local domain R of characteristic p is regular (ie, non-singular) if and only if $F_* R$ is flat as an R-module.

- In our context, this implies that R is smooth if and only if $F_* R$ is locally free.

Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring R would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It's easy to see that $F_* R$ is free of rank p (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

Theorem (Kunz)

A local domain R of characteristic p is regular (ie, non-singular) if and only if $F_* R$ is flat as an R-module.

  - In our context, this implies that R is smooth if and only if $F_* R$ is locally free.

Singularities defined by Frobenius

- How can we use Frobenius to classify singularities?

Definition

A ring R is said to be *F-pure* (or F-split) if there exists a surjective map of R-modules $\phi : F_* R \to R$.

- If $F_* R$ is free as an R-module, it is not hard to see that the Frobenius map splits.

Singularities defined by Frobenius

- How can we use Frobenius to classify singularities?

Definition

A ring R is said to be *F-pure* (or F-split) if there exists a surjective map of R-modules $\phi : F_* R \to R$.

- If $F_* R$ is free as an R-module, it is not hard to see that the Frobenius map splits.

An aside on Frobenius split varieties

- If X is smooth and projective one can still restrict to open sets, corresponding to rings R, which are Frobenius split.
- Therefore, every smooth variety is "locally" Frobenius split.
- However, the the various splittings ϕ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.

An aside on Frobenius split varieties

- If X is smooth and projective one can still restrict to open sets, corresponding to rings R, which are Frobenius split.
- Therefore, every smooth variety is “locally” Frobenius split.
- However, the the various splittings ϕ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.

An aside on Frobenius split varieties

- If X is smooth and projective one can still restrict to open sets, corresponding to rings R, which are Frobenius split.
- Therefore, every smooth variety is “locally” Frobenius split.
- However, the the various splittings ϕ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.

An aside on Frobenius split varieties

- If X is smooth and projective one can still restrict to open sets, corresponding to rings R, which are Frobenius split.
- Therefore, every smooth variety is “locally” Frobenius split.
- However, the the various splittings ϕ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.

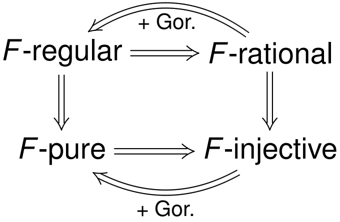
More singularities defined by Frobenius

- Other closely related classes of rings include:
- *(strongly) F-regular*, *F-injective*, and *F-rational*.

- Test ideals (from tight closure theory), F-pure thresholds, and F-pure centers also fit into this framework.

More singularities defined by Frobenius

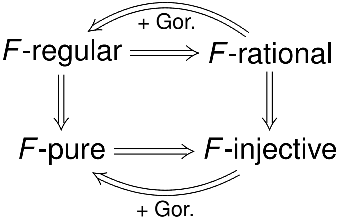
- Other closely related classes of rings include:
- *(strongly) F-regular*, *F-injective*, and *F-rational*.



- Test ideals (from tight closure theory), F-pure thresholds, and F-pure centers also fit into this framework.

More singularities defined by Frobenius

- Other closely related classes of rings include:
- *(strongly) F-regular*, *F-injective*, and *F-rational*.



- Test ideals (from tight closure theory), F-pure thresholds, and F-pure centers also fit into this framework.

Examples of singularities defined by Frobenius

- The ring corresponding to the quadric cone $R = k[x, y, z]/(x^2 + y^2 - z^2)$ is F-regular (except in characteristic 2).
- The ring corresponding to the cubic cone $R = k[x, y, z]/(x^3 + y^3 + z^3)$ is F-pure if and only if $p \equiv 1 \bmod 3$. It is never F-rational.
- The ring corresponding to the node is F-pure but not F-rational.

Examples of singularities defined by Frobenius

- The ring corresponding to the quadric cone $R = k[x, y, z]/(x^2 + y^2 - z^2)$ is F-regular (except in characteristic 2).
- The ring corresponding to the cubic cone $R = k[x, y, z]/(x^3 + y^3 + z^3)$ is F-pure if and only if $p \equiv 1 \bmod 3$. It is never F-rational.
- The ring corresponding to the node is F-pure but not F-rational.

Examples of singularities defined by Frobenius

- The ring corresponding to the quadric cone $R = k[x, y, z]/(x^2 + y^2 - z^2)$ is F-regular (except in characteristic 2).
- The ring corresponding to the cubic cone $R = k[x, y, z]/(x^3 + y^3 + z^3)$ is F-pure if and only if $p \equiv 1 \bmod 3$. It is never F-rational.
- The ring corresponding to the node is F-pure but not F-rational.

Reduction to characteristic $p > 0$

- Suppose you have a complex affine variety X defined by an equation $f(x_1, \ldots, x_n) = 0$.
- If the coefficients of f are integers, then one can also view this as a variety in characteristic $p > 0$.
  - Squint hard, and study the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$
- One says that X has *dense F-pure type* if for infinitely many p, the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ is F-pure.
  - One can similarly define *dense F-regular type*, etc.
- If the coefficients of f are not integers, one can do something similar.

Reduction to characteristic $p > 0$

- Suppose you have a complex affine variety X defined by an equation $f(x_1, \ldots, x_n) = 0$.
- If the coefficients of f are integers, then one can also view this as a variety in characteristic $p > 0$.
  - Squint hard, and study the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$
- One says that X has *dense F-pure type* if for infinitely many p, the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ is F-pure.
  - One can similarly define *dense F-regular type*, etc.
- If the coefficients of f are not integers, one can do something similar.

Reduction to characteristic $p > 0$

- Suppose you have a complex affine variety X defined by an equation $f(x_1, \ldots, x_n) = 0$.
- If the coefficients of f are integers, then one can also view this as a variety in characteristic $p > 0$.
  - Squint hard, and study the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$
- One says that X has *dense F-pure type* if for infinitely many p, the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ is F-pure.
  - One can similarly define *dense F-regular type*, etc.
- If the coefficients of f are not integers, one can do something similar.

Reduction to characteristic $p > 0$

- Suppose you have a complex affine variety X defined by an equation $f(x_1, \ldots, x_n) = 0$.
- If the coefficients of f are integers, then one can also view this as a variety in characteristic $p > 0$.
    - Squint hard, and study the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$
- One says that X has *dense F-pure type* if for infinitely many p, the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ is F-pure.
    - One can similarly define *dense F-regular type*, etc.
- If the coefficients of f are not integers, one can do something similar.

Reduction to characteristic $p > 0$

- Suppose you have a complex affine variety X defined by an equation $f(x_1, \ldots, x_n) = 0$.
- If the coefficients of f are integers, then one can also view this as a variety in characteristic $p > 0$.
    - Squint hard, and study the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$
- One says that X has *dense F-pure type* if for infinitely many p, the ring $\overline{\mathbb{F}_p}[x_1, \ldots, x_n]/(f)$ is F-pure.
    - One can similarly define *dense F-regular type*, etc.
- If the coefficients of f are not integers, one can do something similar.

Reduction to characteristic $p > 0$

- Suppose you have a complex affine variety X defined by an equation $f(x_1, \dots, x_n) = 0$.
- If the coefficients of f are integers, then one can also view this as a variety in characteristic $p > 0$.
  - Squint hard, and study the ring $\overline{\mathbb{F}_p}[x_1, \dots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \dots, x_n]/(f)$
- One says that X has *dense F-pure type* if for infinitely many p, the ring $\overline{\mathbb{F}_p}[x_1, \dots, x_n]/(f)$ is F-pure.
  - One can similarly define *dense F-regular type*, etc.
- If the coefficients of f are not integers, one can do something similar.

Relation between the singularities

- Since about 1980, people have been aware of connections between singularities defined by the action of Frobenius and singularities defined by a resolution of singularities.
- Although the various classes of singularities were introduced independently.
- After the introduction of tight closure by Hochster and Huneke, people began to make the correspondence precise. For example,

Theorem (Smith, Hara/ Mehta-Srinivas)

X has rational singularities if and only if X has dense F-rational type.

Relation between the singularities

- Since about 1980, people have been aware of connections between singularities defined by the action of Frobenius and singularities defined by a resolution of singularities.
- Although the various classes of singularities were introduced independently.
- After the introduction of tight closure by Hochster and Huneke, people began to make the correspondence precise. For example,

Theorem (Smith, Hara/ Mehta-Srinivas)

X has rational singularities if and only if X has dense F-rational type.

Relation between the singularities

- Since about 1980, people have been aware of connections between singularities defined by the action of Frobenius and singularities defined by a resolution of singularities.
- Although the various classes of singularities were introduced independently.
- After the introduction of tight closure by Hochster and Huneke, people began to make the correspondence precise. For example,

Theorem (Smith, Hara/ Mehta-Srinivas)

X has rational singularities if and only if X has dense F-rational type.

More relations between the singularities

- Many other people have since contributed to this dictionary: Fedder, Hara, Mehta, Mustaţă, –, Smith, Srinivas, Takagi, Watanabe, Yoshida and others.

Theorem (–)

If X has dense F-injective type then X has Du Bois singularities.

Theorem (–)

If $W \subseteq X$ is a log canonical center, then after reduction to characteristic $p \gg 0$, $W_p \subseteq X_p$ is a F-pure center.

- Centers of F-purity are very closely related to compatibly Frobenius split subvarieties (which show up often in representation theory).

More relations between the singularities

- Many other people have since contributed to this dictionary: Fedder, Hara, Mehta, Mustaţă, –, Smith, Srinivas, Takagi, Watanabe, Yoshida and others.

Theorem (–)

If X has dense F-injective type then X has Du Bois singularities.

Theorem (–)

If $W \subseteq X$ is a log canonical center, then after reduction to characteristic $p \gg 0$, $W_p \subseteq X_p$ is a F-pure center.

- Centers of F-purity are very closely related to compatibly Frobenius split subvarieties (which show up often in representation theory).

More relations between the singularities

- Many other people have since contributed to this dictionary: Fedder, Hara, Mehta, Mustaţă, –, Smith, Srinivas, Takagi, Watanabe, Yoshida and others.

Theorem (–)

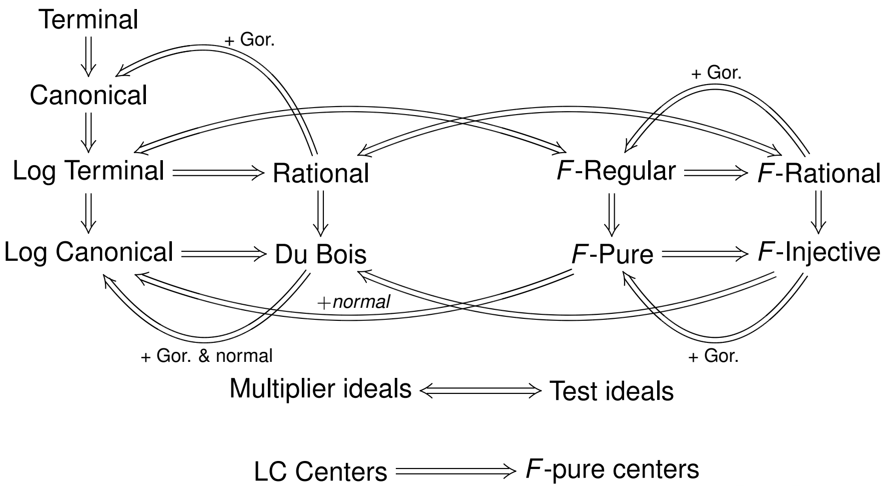
If X has dense F-injective type then X has Du Bois singularities.

Theorem (–)

If $W \subseteq X$ is a log canonical center, then after reduction to characteristic $p \gg 0$, $W_p \subseteq X_p$ is a F-pure center.

- Centers of F-purity are very closely related to compatibly Frobenius split subvarieties (which show up often in representation theory).

The diagram


Terminal
Canonical
Log Terminal
Rational
Log Canonical
Du Bois
+ Gor.
+ Gor.
F-Regular
F-Rational
F-Pure
F-Injective
+normal
+ Gor. & normal
+ Gor.
Multiplier ideals
Test ideals
LC Centers
F-pure centers

Remarks on the diagram

- It is unknown whether there are F-analogues of canonical or terminal singularities.
- It is conjectured that log canonical singularities are of dense F-pure type, but this is (very) open.
- Of course, this diagram has been used to inspire questions in both contexts. It has also been used to answer questions.

Remarks on the diagram

- It is unknown whether there are F-analogues of canonical or terminal singularities.
- It is conjectured that log canonical singularities are of dense F-pure type, but this is (very) open.
- Of course, this diagram has been used to inspire questions in both contexts. It has also been used to answer questions.

Remarks on the diagram

- It is unknown whether there are F-analogues of canonical or terminal singularities.
- It is conjectured that log canonical singularities are of dense F-pure type, but this is (very) open.
- Of course, this diagram has been used to inspire questions in both contexts. It has also been used to answer questions.

Multiplier ideals

- Suppose X is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases t, these become smaller ideals.

$$\mathcal{J}(X, \mathfrak{a}^{t_1}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_2}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_3}) \supsetneq \dots$$

- They change at a discrete set of rational numbers t_i, called *jumping numbers*.
  - At least when X is normal and $\mathbb{Q}$-Gorenstein.

Multiplier ideals

- Suppose X is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases t, these become smaller ideals.

$$\mathcal{J}(X, \mathfrak{a}^{t_1}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_2}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_3}) \supsetneq \dots$$

- They change at a discrete set of rational numbers t_i, called *jumping numbers*.
  - At least when X is normal and $\mathbb{Q}$-Gorenstein.

Multiplier ideals

- Suppose X is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases t, these become smaller ideals.

$$\mathcal{J}(X, \mathfrak{a}^{t_1}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_2}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_3}) \supsetneq \dots$$

- They change at a discrete set of rational numbers t_i, called *jumping numbers*.
  - At least when X is normal and $\mathbb{Q}$-Gorenstein.

Multiplier ideals

- Suppose X is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases t, these become smaller ideals.

$$\mathcal{J}(X, \mathfrak{a}^{t_1}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_2}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_3}) \supsetneq \dots$$

- They change at a discrete set of rational numbers t_i, called *jumping numbers*.
  - At least when X is normal and $\mathbb{Q}$-Gorenstein.

Multiplier ideals

- Suppose X is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases t, these become smaller ideals.

$$\mathcal{J}(X, \mathfrak{a}^{t_1}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_2}) \supsetneq \mathcal{J}(X, \mathfrak{a}^{t_3}) \supsetneq \dots$$

- They change at a discrete set of rational numbers t_i, called *jumping numbers*.
    - At least when X is normal and $\mathbb{Q}$-Gorenstein.

Frobenius jumping numbers

- But the test ideal, $\tau(X, \mathfrak{a}^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same "jumping" behavior holds (for a fixed $\mathfrak{a}$).

Theorem (Blickle, Mustaţă, Smith)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is smooth.

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

Theorem (–, Takagi, Zhang)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is normal and $\mathbb{Q}$-Gorenstein with index not divisible by p.

Frobenius jumping numbers

- But the test ideal, $\tau(X, \mathfrak{a}^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same "jumping" behavior holds (for a fixed $\mathfrak{a}$).

Theorem (Blickle, Mustaţă, Smith)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is smooth.

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

Theorem (–, Takagi, Zhang)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is normal and $\mathbb{Q}$-Gorenstein with index not divisible by p.

Frobenius jumping numbers

- But the test ideal, $\tau(X, \mathfrak{a}^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same "jumping" behavior holds (for a fixed $\mathfrak{a}$).

Theorem (Blickle, Mustaţă, Smith)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is smooth.

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

Theorem (–, Takagi, Zhang)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is normal and $\mathbb{Q}$-Gorenstein with index not divisible by p.

Frobenius jumping numbers

- But the test ideal, $\tau(X, \mathfrak{a}^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same "jumping" behavior holds (for a fixed $\mathfrak{a}$).

Theorem (Blickle, Mustaţă, Smith)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is smooth.

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

Theorem (–, Takagi, Zhang)

The set of "F-jumping numbers" for $\mathfrak{a}$ are discrete and rational when X is normal and $\mathbb{Q}$-Gorenstein with index not divisible by p.