Outline

1. Singularities on algebraic varieties
   - Algebraic varieties
   - Singularities

2. Types of singularities in characteristic zero
   - Resolution of singularities
   - Classifying singularities using resolutions

3. Singularities in characteristic $p > 0$
   - Definitions
   - Characteristic 0 vs characteristic $p > 0$ singularities
Outline

1. Singularities on algebraic varieties
   - Algebraic varieties
   - Singularities

2. Types of singularities in characteristic zero
   - Resolution of singularities
   - Classifying singularities using resolutions

3. Singularities in characteristic $p > 0$
   - Definitions
   - Characteristic 0 vs characteristic $p > 0$ singularities
Affine algebraic varieties

- What is a complex affine algebraic variety?
  - It is a subset of \( \mathbb{C}^n \) which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I’ll only consider varieties defined by a single equation (hypersurfaces).

- For example, in \( \mathbb{C}^2 \) one might consider \( y - x^2 \) or \( y^2 - x^3 \) or \( y^2 - x^2(x - 1) \).

- Of course, these are two dimensional objects really, we only plotted their real points.
What is a complex affine algebraic variety?
- It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
- In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x - 1)$.
- Of course, these are two dimensional objects really, we only plotted their real points.
Affine algebraic varieties

- What is a complex affine algebraic variety?
  - It is a subset of \( \mathbb{C}^n \) which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I'll only consider varieties defined by a single equation (hypersurfaces).

- For example, in \( \mathbb{C}^2 \) one might consider \( y - x^2 \) or \( y^2 - x^3 \) or \( y^2 - x^2(x - 1) \).

- Of course, these are two dimensional objects really, we only plotted their real points.
Affine algebraic varieties

- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I’ll only consider varieties defined by a single equation (hypersurfaces).

- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x - 1)$.

- Of course, these are two dimensional objects really, we only plotted their real points.
Affine algebraic varieties

- What is a complex affine algebraic variety?
  - It is a subset of $\mathbb{C}^n$ which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I’ll only consider varieties defined by a single equation (hypersurfaces).
- For example, in $\mathbb{C}^2$ one might consider $y - x^2$ or $y^2 - x^3$ or $y^2 - x^2(x - 1)$.
- Of course, these are two dimensional objects really, we only plotted their real points.
Affine algebraic varieties

- What is a complex affine algebraic variety?
  - It is a subset of \( \mathbb{C}^n \) which is the vanishing set of some collection of polynomial equations.
  - In the examples of this talk, I’ll only consider varieties defined by a single equation (hypersurfaces).

- For example, in \( \mathbb{C}^2 \) one might consider \( y - x^2 \) or \( y^2 - x^3 \) or \( y^2 - x^2(x - 1) \).

- Of course, these are two dimensional objects really, we only plotted their real points.
Higher dimensional examples I

- In $\mathbb{C}^3$ one might consider a quadric cone, $x^2 + y^2 - z^2$.

- Or a cone over a cubic, $y^2 z - x(x - z)(x + z)$. 
Higher dimensional examples I

- In $\mathbb{C}^3$ one might consider a quadric cone, $x^2 + y^2 - z^2$.

- Or a cone over a cubic, $y^2z - x(x - z)(x + z)$.
These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).

- Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_\mathbb{C}$.

- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.

- There won’t be any positive characteristic drawings.
Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of \( \mathbb{P}^n_\mathbb{C} \).
  
- We also work over other fields besides \( \mathbb{C} \). In particular, sometimes we work over fields of characteristic \( p > 0 \).
  
- There won’t be any positive characteristic drawings.
Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}_C^n$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$. There won’t be any positive characteristic drawings.
Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_{\mathbb{C}}$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.
  - There won’t be any positive characteristic drawings.
Generalizations

- These examples are not compact (they are affine). Often one studies “projective” algebraic varieties (which are compact).
  - Projective algebraic varieties are simply several affine algebraic varieties glued together (on large open patches) in such a way that they embed algebraically as a closed subset of $\mathbb{P}^n_C$.
- We also work over other fields besides $\mathbb{C}$. In particular, sometimes we work over fields of characteristic $p > 0$.
  - There won’t be any positive characteristic drawings.
Relation with algebra

- If one is studying a complex affine variety \( X \) defined by an equation \( f(x_1, \ldots, x_n) = 0 \), the ring

\[
R = \mathbb{C}[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))
\]

carries the same information as \( X \) (although it doesn’t record the embedding \( X \subseteq \mathbb{C}^n \)).

- The points of the variety correspond to the maximal ideals of the ring \( R \).

- Therefore, one can study the algebraic variety \( X \) by studying the ring \( R \).

- This is particularly useful when working over fields besides \( \mathbb{C} \).
Relation with algebra

- If one is studying a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$, the ring
  $$R = \mathbb{C}[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$$
  carries the same information as $X$ (although it doesn’t record the embedding $X \subseteq \mathbb{C}^n$).
- The points of the variety correspond to the maximal ideals of the ring $R$.
- Therefore, one can study the algebraic variety $X$ by studying the ring $R$.
- This is particularly useful when working over fields besides $\mathbb{C}$. 
Relation with algebra

- If one is studying a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$, the ring
  \[ R = \mathbb{C}[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n)) \]
carries the same information as $X$ (although it doesn’t record the embedding $X \subseteq \mathbb{C}^n$).
- The points of the variety correspond to the maximal ideals of the ring $R$.
- Therefore, one can study the algebraic variety $X$ by studying the ring $R$.
- This is particularly useful when working over fields besides $\mathbb{C}$. 

Karl Schwede
Relation with algebra

- If one is studying a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$, the ring
  $$R = \mathbb{C}[x_1, \ldots, x_n]/(f(x_1, \ldots, x_n))$$
carries the same information as $X$ (although it doesn’t record the embedding $X \subseteq \mathbb{C}^n$).

- The points of the variety correspond to the maximal ideals of the ring $R$.

- Therefore, one can study the algebraic variety $X$ by studying the ring $R$.

- This is particularly useful when working over fields besides $\mathbb{C}$.
This talk is about singularities... so

- **What is a singularity?**
  - On a complex variety, a point $Q$ is *smooth* if “very locally”, that point looks the same as a point of $\mathbb{C}^d$.
  - A point is *singular* if it is not smooth.
  - Alternately, if $X$ is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point $Q$ is singular if $f(Q) = 0$ and $\frac{\partial f}{\partial x_i}(Q) = 0$ for each $i = 1, \ldots, n$.
    - This description works also when working over other fields.
    - One can do something similar for non-hypersurfaces.
  
- All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.
This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point $Q$ is \textit{smooth} if “very locally”, that point looks the same as a point of $\mathbb{C}^d$.
- A point is \textit{singular} if it is not smooth.
- Alternately, if $X$ is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point $Q$ is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.
This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point $Q$ is *smooth* if “very locally”, that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if $X$ is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point $Q$ is singular if $f(Q) = 0$ and $\partial f / \partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.

- All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.
This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point $Q$ is *smooth* if “very locally”, that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if $X$ is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point $Q$ is singular if $f(Q) = 0$ and $\frac{\partial f}{\partial x_i}(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.

All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.
This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point \( Q \) is \textit{smooth} if “very locally”, that point looks the same as a point of \( \mathbb{C}^d \).
- A point is \textit{singular} if it is not smooth.
- Alternately, if \( X \) is defined by a single equation \( f(x_1, \ldots, x_n) = 0 \), then a point \( Q \) is singular if \( f(Q) = 0 \) and \( \frac{\partial f}{\partial x_i}(Q) = 0 \) for each \( i = 1, \ldots, n \).
- This description works also when working over other fields.
- One can do something similar for non-hypersurfaces.
- All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.
This talk is about singularities... so

- **What is a singularity?**
- **On a complex variety, a point** $Q$ **is** smooth **if “very locally”, that point looks the same as a point of** $\mathbb{C}^d$.
- **A point is** singular **if it is not smooth.**
- **Alternately, if** $X$ **is defined by a single equation**
  
  $$f(x_1, \ldots, x_n) = 0,$$

  **then a point** $Q$ **is singular if** $f(Q) = 0$ **and**
  
  $$\frac{\partial f}{\partial x_i}(Q) = 0$$

  **for each** $i = 1, \ldots, n$.

  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.

- All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.
This talk is about singularities... so

- What is a singularity?
- On a complex variety, a point $Q$ is *smooth* if “very locally”, that point looks the same as a point of $\mathbb{C}^d$.
- A point is *singular* if it is not smooth.
- Alternately, if $X$ is defined by a single equation $f(x_1, \ldots, x_n) = 0$, then a point $Q$ is singular if $f(Q) = 0$ and $\partial f/\partial x_i(Q) = 0$ for each $i = 1, \ldots, n$.
  - This description works also when working over other fields.
  - One can do something similar for non-hypersurfaces.
- All the examples we’ve looked at so far (except the parabola) have an “isolated singularity” at the origin.
Why study singularities? I

Perhaps you are only interested in smooth varieties?

- Singularities show up as limits of smooth varieties.

  This happens particularly when “compactifying moduli spaces”
  (moduli spaces are algebraic varieties whose points parameterize something. For example, points can correspond to isomorphism classes of certain varieties).
Why study singularities? I

Perhaps you are only interested in smooth varieties?

- Singularities show up as limits of smooth varieties.

- This happens particularly when “compactifying moduli spaces”
  (moduli spaces are algebraic varieties whose points parameterize something. For example, points can correspond to isomorphism classes of certain varieties).
Why study singularities? I

Perhaps you are only interested in smooth varieties?

- Singularities show up as limits of smooth varieties.

- This happens particularly when “compactifying moduli spaces”
  - (moduli spaces are algebraic varieties whose points parameterize something. For example, points can correspond to isomorphism classes of certain varieties).
Why study singularities? II

- If you want to classify algebraic varieties, sometimes you need to replace a variety $X$ with a simpler but closely related variety $Y$.
- One way in which this is done is by contracting (compact) subsets of varieties to points.
- This happens in the minimal model program.
Why study singularities? II

○ If you want to classify algebraic varieties, sometimes you need to replace a variety $X$ with a simpler but closely related variety $Y$.

○ One way in which this is done is by contracting (compact) subsets of varieties to points.

This happens in the minimal model program.
Why study singularities? II

- If you want to classify algebraic varieties, sometimes you need to replace a variety $X$ with a simpler but closely related variety $Y$.
- One way in which this is done is by contracting (compact) subsets of varieties to points.
- This happens in the minimal model program.
Of course, sometimes you simply want to generalize a theorem to as broad a setting as possible, and so you ask: “What property of smooth varieties allows me to prove this theorem?”

Once you can answer this question, you have identified a class of singularities.
Why study singularities? III

- Of course, sometimes you simply want to generalize a theorem to as broad a setting as possible, and so you ask “What property of smooth varieties allows me to prove this theorem?”
- Once you can answer this question, you have identified a class of singularities.
Why study singularities? III

- Of course, sometimes you simply want to generalize a theorem to as broad a setting as possible, and so you ask “What property of smooth varieties allows me to prove this theorem?”
- Once you can answer this question, you have identified a class of singularities.
Outline

1. Singularities on algebraic varieties
   - Algebraic varieties
   - Singularities

2. Types of singularities in characteristic zero
   - Resolution of singularities
   - Classifying singularities using resolutions

3. Singularities in characteristic $p > 0$
   - Definitions
   - Characteristic 0 vs characteristic $p > 0$ singularities
What is a resolution of singularities?

Suppose you are given a singular variety $X$.

A *resolution of singularities* is a map of algebraic varieties $\pi : \tilde{X} \to X$ that satisfies the following properties:

- $\tilde{X}$ is smooth.
- $\pi$ is “birational” (this means it is an isomorphism outside of a small closed subset of $X$, usually the singular locus of $X$).
- $\pi$ is “proper” (in particular, this implies that the pre-image of a point is compact).
- Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
- We also usually require that the pre-image of the singular locus looks like “coordinate hyperplanes”, sufficiently locally.

Resolutions of singularities always exist in characteristic zero.
What is a resolution of singularities?

- Suppose you are given a singular variety $X$.
- A *resolution of singularities* is a map of algebraic varieties
  $\pi : \tilde{X} \to X$ that satisfies the following properties:
    - $\tilde{X}$ is smooth.
    - $\pi$ is “birational” (this means it is an isomorphism outside of
      a small closed subset of $X$, usually the singular locus of $X$)
    - $\pi$ is “proper” (in particular, this implies that the pre-image of
      a point is compact)
      - Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
    - We also usually require that the pre-image of the singular
      locus looks like “coordinate hyperplanes”, sufficiently
      locally.
- Resolutions of singularities always exist in characteristic zero.
What is a resolution of singularities?

- Suppose you are given a singular variety $X$.
- A \textit{resolution of singularities} is a map of algebraic varieties $\pi : \tilde{X} \to X$ that satisfies the following properties:
  - $\tilde{X}$ is smooth.
  - $\pi$ is “birational” (this means it is an isomorphism outside of a small closed subset of $X$, usually the singular locus of $X$)
  - $\pi$ is “proper” (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
  - We also usually require that the pre-image of the singular locus looks like “coordinate hyperplanes”, sufficiently locally.
- Resolutions of singularities always exist in characteristic zero.
What is a resolution of singularities?

- Suppose you are given a singular variety $X$.
- A **resolution of singularities** is a map of algebraic varieties $\pi : \tilde{X} \to X$ that satisfies the following properties:
  - $\tilde{X}$ is smooth.
  - $\pi$ is “birational” (this means it is an isomorphism outside of a small closed subset of $X$, usually the singular locus of $X$)
  - $\pi$ is “proper” (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
  - We also usually require that the pre-image of the singular locus looks like “coordinate hyperplanes”, sufficiently locally.
- Resolutions of singularities always exist in characteristic zero.
What is a resolution of singularities?

- Suppose you are given a singular variety $X$.
- A *resolution of singularities* is a map of algebraic varieties $\pi: \tilde{X} \to X$ that satisfies the following properties:
  - $\tilde{X}$ is smooth.
  - $\pi$ is “birational” (this means it is an isomorphism outside of a small closed subset of $X$, usually the singular locus of $X$)
  - $\pi$ is “proper” (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
    - We also usually require that the pre-image of the singular locus looks like “coordinate hyperplanes”, sufficiently locally.

- Resolutions of singularities always exist in characteristic zero
What is a resolution of singularities?

- Suppose you are given a singular variety $X$.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \tilde{X} \to X$ that satisfies the following properties:
  - $\tilde{X}$ is smooth.
  - $\pi$ is “birational” (this means it is an isomorphism outside of a small closed subset of $X$, usually the singular locus of $X$)
  - $\pi$ is “proper” (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
  - We also usually require that the pre-image of the singular locus looks like “coordinate hyperplanes”, sufficiently locally.
- Resolutions of singularities always exist in characteristic zero.
What is a resolution of singularities?

- Suppose you are given a singular variety $X$.
- A *resolution of singularities* is a map of algebraic varieties $\pi : \tilde{X} \to X$ that satisfies the following properties:
  - $\tilde{X}$ is smooth.
  - $\pi$ is “birational” (this means it is an isomorphism outside of a small closed subset of $X$, usually the singular locus of $X$)
  - $\pi$ is “proper” (in particular, this implies that the pre-image of a point is compact)
    - Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
  - We also usually require that the pre-image of the singular locus looks like “coordinate hyperplanes”, sufficiently locally.

- Resolutions of singularities always exist in characteristic zero.
What is a resolution of singularities?

- Suppose you are given a singular variety $X$.
- A \textit{resolution of singularities} is a map of algebraic varieties $\pi : \tilde{X} \to X$ that satisfies the following properties:
  - $\tilde{X}$ is smooth.
  - $\pi$ is “birational” (this means it is an isomorphism outside of a small closed subset of $X$, usually the singular locus of $X$).
  - $\pi$ is “proper” (in particular, this implies that the pre-image of a point is compact).
    - Because of this, $\tilde{X}$ is usually not affine, even when $X$ is.
  - We also usually require that the pre-image of the singular locus looks like “coordinate hyperplanes”, sufficiently locally.
- Resolutions of singularities always exist in characteristic zero.
Why resolve singularities?

- A resolution of singularities takes your variety $X$ and constructs a “smooth variety” $\tilde{X}$ that is very closely related to $X$.
  - $\tilde{X}$ and $X$ are “birational”.
- The “properness” of the resolution implies that if $X$ was compact, then $\tilde{X}$ is also compact.
- So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.
Why resolve singularities?

- A resolution of singularities takes your variety $X$ and constructs a “smooth variety” $\tilde{X}$ that is very closely related to $X$.
  - $\tilde{X}$ and $X$ are “birational”.
- The “properness” of the resolution implies that if $X$ was compact, then $\tilde{X}$ is also compact.
- So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.
Why resolve singularities?

- A resolution of singularities takes your variety $X$ and constructs a “smooth variety” $\tilde{X}$ that is very closely related to $X$.
  - $\tilde{X}$ and $X$ are “birational”.
- The “properness” of the resolution implies that if $X$ was compact, then $\tilde{X}$ is also compact.
- So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.
Why resolve singularities?

• A resolution of singularities takes your variety $X$ and constructs a “smooth variety” $\tilde{X}$ that is very closely related to $X$.
  • $\tilde{X}$ and $X$ are “birational”.

• The “properness” of the resolution implies that if $X$ was compact, then $\tilde{X}$ is also compact.

• So sometimes if you know a theorem about smooth varieties, you can prove the same theorem about singular varieties just by using this resolution.
How do you resolve singularities?

- You perform several blow-ups.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

Theorem (Hironaka)

*In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities.*
How do you resolve singularities?

- You perform several *blow-ups*.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

*Theorem (Hironaka)*

*In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities*
How do you resolve singularities?

- You perform several *blow-ups*.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

**Theorem (Hironaka)**

*In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities.*
How do you resolve singularities?

- You perform several *blow-ups*.
  - A blow-up is an “un-contraction” of a closed subset.
  - It is exactly the opposite operation of the example from before.

**Theorem (Hironaka)**

*In characteristic zero, if you do enough blow-ups at “smooth centers”, in the right order, you will construct a resolution of singularities*
Example with curves

- We will blow-up points in \( \mathbb{C}^2 \) and see what it does to curves.
- A blow-up at a point on \( \mathbb{C}^2 \) turns every different tangent direction (discounting sign) at \( Q \) into its own point. It replaces \( Q \) by a copy of \( \mathbb{P}^1_\mathbb{C} = \text{"The Riemann sphere"} \).
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.

The black line is the \( \mathbb{P}^1_\mathbb{C} \) that will be contracted back to the origin in \( \mathbb{C}^2 \).
Example with curves

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at $Q$ into its own point. It replaces $Q$ by a copy of $\mathbb{P}^1_C = \text{“The Riemann sphere”}$. 
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.

The black line is the $\mathbb{P}^1_C$ that will be contracted back to the origin in $\mathbb{C}^2$. 

blown-up becomes

Karl Schwede
**Example with curves**

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at $Q$ into its own point. It replaces $Q$ by a copy of $\mathbb{P}^1_{\mathbb{C}}$ = “The Riemann sphere”.
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.

The black line is the $\mathbb{P}^1_{\mathbb{C}}$ that will be contracted back to the origin in $\mathbb{C}^2$. The blown-up becomes
Example with curves

- We will blow-up points in $\mathbb{C}^2$ and see what it does to curves.
- A blow-up at a point on $\mathbb{C}^2$ turns every different tangent direction (discounting sign) at $Q$ into its own point. It replaces $Q$ by a copy of $\mathbb{P}^1_{\mathbb{C}} = \text{"The Riemann sphere"}$. 
- What happens to curves on the plane? 
- This separation of tangent directions means that nodes become separated.

The black line is the $\mathbb{P}^1_{\mathbb{C}}$ that will be contracted back to the origin in $\mathbb{C}^2$. 

blown-up becomes
Singularities on algebraic varieties

Types of singularities in characteristic zero

Singularities in characteristic \( p > 0 \)

Resolution of singularities

Classifying singularities using resolutions

Example with curves

- We will blow-up points in \( \mathbb{C}^2 \) and see what it does to curves.
- A blow-up at a point on \( \mathbb{C}^2 \) turns every different tangent direction (discounting sign) at \( Q \) into its own point. It replaces \( Q \) by a copy of \( \mathbb{P}^1_C = \text{“The Riemann sphere”} \).
- What happens to curves on the plane?
- This separation of tangent directions means that nodes become separated.

blown-up becomes

- The black line is the \( \mathbb{P}^1_C \) that will be contracted back to the origin in \( \mathbb{C}^2 \).
A similar thing happens with the quadric cones in $\mathbb{C}^3$.

When we do the blow-up at the origin, all the different tangent directions get separated. But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle. At least its real points look like a circle.
Additional discussion of blow-ups

- A similar thing happens with the quadric cones in $\mathbb{C}^3$.

![Diagram of blow-up]

- When we do the blow-up at the origin, all the different tangent directions get separated.
  - But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle.
  - At least its real points look like a circle.
A similar thing happens with the quadric cones in $\mathbb{C}^3$.

When we do the blow-up at the origin, all the different tangent directions get separated.

But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle.

at least its real points look like a circle.
Additional discussion of blow-ups

- A similar thing happens with the quadric cones in $\mathbb{C}^3$.

- When we do the blow-up at the origin, all the different tangent directions get separated.
- But this just replaces the singular point of the cone with the distinct tangent directions that go into it, in this case with a circle.
  - at least its real points look like a circle.
How can we classify singularities with resolutions?

- All the examples we’ve seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.

- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).

- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different “minimal” ways to resolve the same singularity.

- You can often compare the (geometric / algebraic / homological) properties of the resolution $\tilde{X}$ with those same (geometric / algebraic / homological) properties of $X$. 
How can we classify singularities with resolutions?

- All the examples we’ve seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.
- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).
- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different “minimal” ways to resolve the same singularity.
- You can often compare the (geometric / algebraic / homological) properties of the resolution $\tilde{X}$ with those same (geometric / algebraic / homological) properties of $X$. 
How can we classify singularities with resolutions?

- All the examples we’ve seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.
- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).
- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different “minimal” ways to resolve the same singularity.
- You can often compare the (geometric / algebraic / homological) properties of the resolution $\tilde{X}$ with those same (geometric / algebraic / homological) properties of $X$. 
How can we classify singularities with resolutions?

- All the examples we’ve seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.

- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).

- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different “minimal” ways to resolve the same singularity.
  - You can often compare the (geometric / algebraic / homological) properties of the resolution $\tilde{X}$ with those same (geometric / algebraic / homological) properties of $X$. 
How can we classify singularities with resolutions?

- All the examples we’ve seen so far can be resolved by one blow-up at a single point. However, there are many singularities that require more work to resolve.

- One option then is to study the (minimal) blow-ups needed to resolve the singularities.
  - You can do something like this for surfaces (surface = 2 complex dimensions).

- However, in higher dimensions this becomes difficult (and also much harder to visualize). There are also different “minimal” ways to resolve the same singularity.

- You can often compare the (geometric / algebraic / homological) properties of the resolution $\tilde{X}$ with those same (geometric / algebraic / homological) properties of $X$. 
The goal of the minimal model program is to take a “birational equivalence class” of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with “mild” singularities.

What does mild mean? One compares the sheaf of “top dimensional differentials” on $X$ (naively extended over the singular locus) with the top differentials of its resolution $\tilde{X}$.

Singularities classified this way behave well with respect to the contractions of the minimal model program.

Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.
The goal of the minimal model program is to take a “birational equivalence class” of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with “mild” singularities.

What does mild mean? One compares the sheaf of “top dimensional differentials” on $X$ (naively extended over the singular locus) with the top differentials of its resolution $\tilde{X}$.

Singularities classified this way behave well with respect to the contractions of the minimal model program.

Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.
Singularities of the minimal model program I

- The goal of the minimal model program is to take a “birational equivalence class” of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with “mild” singularities.

- What does mild mean? One compares the sheaf of “top dimensional differentials” on $X$ (naively extended over the singular locus) with the top differentials of its resolution $\tilde{X}$.

- Singularities classified this way behave well with respect to the contractions of the minimal model program.

- Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.
The goal of the minimal model program is to take a “birational equivalence class” of varieties and find a good minimal representative of that class. In particular, one contracts certain closed subvarieties in order to get new varieties with “mild” singularities.

What does mild mean? One compares the sheaf of “top dimensional differentials” on $X$ (naively extended over the singular locus) with the top differentials of its resolution $\tilde{X}$.

Singularities classified this way behave well with respect to the contractions of the minimal model program.

Certain important theorems (such as the Kodaira vanishing theorem) also hold on varieties with these singularities.
Recall we are defining singularities by looking at how the sheaf of top differential forms on a resolution $\tilde{X}$ behaves compared to the sheaf of top differentials on $X$.

By looking at the numerics of these comparisons, one can write down definitions of *terminal*, *canonical*, *log terminal*, *log canonical*, *rational* and *Du Bois* singularities.

Actually, Du Bois singularities were originally defined using other methods (Hodge Theory), although we now have the following theorem.

**Theorem (Kovács, –, Smith)**

Suppose that $X$ is normal and Cohen-Macaulay and $\pi : \tilde{X} \to X$ is a (log) resolution of $X$ with exceptional set $E$. Then $X$ has Du Bois singularities if and only if $\pi_* \omega_{\tilde{X}}(E) = \omega_X$. 
Recall we are defining singularities by looking at how the sheaf of top differential forms on a resolution $\tilde{X}$ behaves compared to the sheaf of top differentials on $X$.

By looking at the numerics of these comparisons, one can write down definitions of terminal, canonical, log terminal, log canonical, rational and Du Bois singularities.

Actually, Du Bois singularities were originally defined using other methods (Hodge Theory), although we now have the following theorem.

**Theorem (Kovács, –, Smith)**

Suppose that $X$ is normal and Cohen-Macaulay and $\pi : \tilde{X} \to X$ is a (log) resolution of $X$ with exceptional set $E$. Then $X$ has Du Bois singularities if and only if $\pi_*\omega_{\tilde{X}}(E) = \omega_X$. 

Karl Schwede
Recall we are defining singularities by looking at how the sheaf of top differential forms on a resolution $\tilde{X}$ behaves compared to the sheaf of top differentials on $X$.

By looking at the numerics of these comparisons, one can write down definitions of *terminal, canonical, log terminal, log canonical, rational* and *Du Bois* singularities.

Actually, Du Bois singularities were originally defined using other methods (Hodge Theory), although we now have the following theorem.

**Theorem (Kovács, –, Smith)**

*Suppose that $X$ is normal and Cohen-Macaulay and $\pi : \tilde{X} \to X$ is a (log) resolution of $X$ with exceptional set $E$. Then $X$ has Du Bois singularities if and only if $\pi_* \omega_{\tilde{X}}(E) = \omega_X$.*
Singularities of the minimal model program III

- The following diagram summarizes implications between the singularities of the minimal model program.

  ![Diagram](image)

- Not all of the implications in the above diagram are trivial, see the work of Elkik, Ishii, Kollár, Kovács, Saito, –, Smith, Steenbrink and others.

- Multiplier ideals, adjoint ideals, log canonical thresholds and log canonical centers are also measures of singularities that fit into the same framework.
The following diagram summarizes implications between the singularities of the minimal model program.

```
Terminal → Canonical → Log Terminal → Rational

↓ ↓
Log Canonical → Du Bois
```

Not all of the implications in the above diagram are trivial, see the work of Elkik, Ishii, Kollár, Kovács, Saito, –, Smith, Steenbrink and others.

Multiplier ideals, adjoint ideals, log canonical thresholds and log canonical centers are also measures of singularities that fit into the same framework.
Singularities of the minimal model program III

- The following diagram summarizes implications between the singularities of the minimal model program.

```
Terminal → Canonical → Log Terminal → Rational
↓↓↓↓
Log Canonical → Du Bois
```

- Not all of the implications in the above diagram are trivial, see the work of Elkik, Ishii, Kollár, Kovács, Saito, –, Smith, Steenbrink and others.

- Multiplier ideals, adjoint ideals, log canonical thresholds and log canonical centers are also measures of singularities that fit into the same framework.
Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.
Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.
Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.
Our examples

- The quadric cone we discussed is canonical but not terminal.
- The cubic cone is log canonical but not rational.
- The nodal curve is only Du Bois.
- The cuspidal curve is not even Du Bois.
There are analytic ways to describe several of the singularities of the minimal model program as well.

For example, consider a variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.

Also assume that $f$ is irreducible.

Then $X$ is (semi) log canonical near the origin 0 if and only if

$$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}}$$

is integrable near 0 for all $c < 1$.

The multiplier ideal can also be described in a similar way.
Analytic description of singularities

- There are analytic ways to describe several of the singularities of the minimal model program as well.
- For example, consider a variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.
- Also assume that $f$ is irreducible.
- Then $X$ is (semi) log canonical near the origin $0$ if and only if

$$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}}$$

is integrable near $0$ for all $c < 1$.
- The multiplier ideal can also be described in a similar way.
There are analytic ways to describe several of the singularities of the minimal model program as well.

For example, consider a variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.

Also assume that $f$ is irreducible.

Then $X$ is (semi) log canonical near the origin 0 if and only if

$$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}}$$

is integrable near 0 for all $c < 1$.

The multiplier ideal can also be described in a similar way.
There are analytic ways to describe several of the singularities of the minimal model program as well.

For example, consider a variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.

Also assume that $f$ is irreducible.

Then $X$ is (semi) log canonical near the origin $0$ if and only if

$$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}}$$

is integrable near $0$ for all $c < 1$.

The multiplier ideal can also be described in a similar way.
Analytic description of singularities

- There are analytic ways to describe several of the singularities of the minimal model program as well.
- For example, consider a variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$ in $\mathbb{C}^n$.
- Also assume that $f$ is irreducible.
- Then $X$ is (semi) log canonical near the origin $0$ if and only if
  $$\frac{1}{|f(x_1, \ldots, x_n)|^{2c}}$$
  is integrable near $0$ for all $c < 1$.
- The multiplier ideal can also be described in a similar way.
Outline

1. Singularities on algebraic varieties
   - Algebraic varieties
   - Singularities

2. Types of singularities in characteristic zero
   - Resolution of singularities
   - Classifying singularities using resolutions

3. Singularities in characteristic \( p > 0 \)
   - Definitions
   - Characteristic 0 vs characteristic \( p > 0 \) singularities
What’s different about characteristic $p$?

- Suppose that $k$ is an algebraically closed field of characteristic $p$.
- One can still make sense of varieties defined over $k$.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone’s satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic $p$. 

Karl Schwede
What’s different about characteristic $p$?

- Suppose that $k$ is an algebraically closed field of characteristic $p$.
- One can still make sense of varieties defined over $k$.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone’s satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic $p$. 
What’s different about characteristic $p$?

- Suppose that $k$ is an algebraically closed field of characteristic $p$.
- One can still make sense of varieties defined over $k$.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone’s satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic $p$. 
What’s different about characteristic $p$?

- Suppose that $k$ is an algebraically closed field of characteristic $p$.
- One can still make sense of varieties defined over $k$.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone’s satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic $p$. 
What’s different about characteristic $p$?

- Suppose that $k$ is an algebraically closed field of characteristic $p$.
- One can still make sense of varieties defined over $k$.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone’s satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic $p$. 

Karl Schwede
What’s different about characteristic $p$?

- Suppose that $k$ is an algebraically closed field of characteristic $p$.
- One can still make sense of varieties defined over $k$.
- Singularities can even still be detected using partial derivatives.
- Resolution of singularities is still an open question at this point.
  - Although there is hope that this might be solved to everyone’s satisfaction shortly.
- However, some technical (vanishing) theorems used to prove properties of singularities are known to be false in characteristic $p$. 
Various people have been studying properties of rings in characteristic $p > 0$ for a long time.

Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.

The Frobenius map on a ring $R$ is the map $F : R \to R$ that sends $x \in R$ to $x^p$ (where $p$ is the characteristic of $R$).

Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.

If $R$ is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R$$ or the inclusion $R \subset R^{1/p}$. 
Various people have been studying properties of rings in characteristic $p > 0$ for a long time.

Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.

The Frobenius map on a ring $R$ is the map $F : R \rightarrow R$ that sends $x \in R$ to $x^p$ (where $p$ is the characteristic of $R$).

Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.

If $R$ is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R$$

or the inclusion $R \subset R^{1/p}$.
Various people have been studying properties of rings in characteristic $p > 0$ for a long time.

Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.

- The Frobenius map on a ring $R$ is the map $F : R \rightarrow R$ that sends $x \in R$ to $x^p$ (where $p$ is the characteristic of $R$).

- Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.

- If $R$ is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.$$
Various people have been studying properties of rings in characteristic $p > 0$ for a long time.

Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.

The Frobenius map on a ring $R$ is the map $F : R \to R$ that sends $x \in R$ to $x^p$ (where $p$ is the characteristic of $R$).

Frobenius is a ring homomorphism since $(x + y)^p = x^p + y^p$.

If $R$ is reduced (there are no elements $0 \neq x \in R$ such that $x^p = 0$), then the Frobenius map can be thought of as the inclusion:

$$R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.$$
Various people have been studying properties of rings in characteristic \( p > 0 \) for a long time.

Algebraic geometers and commutative algebraists have classified singularities of these rings by studying the action of Frobenius.

- The Frobenius map on a ring \( R \) is the map \( F : R \to R \) that sends \( x \in R \) to \( x^p \) (where \( p \) is the characteristic of \( R \)).

Frobenius is a ring homomorphism since 
\[
(x + y)^p = x^p + y^p.
\]

If \( R \) is reduced (there are no elements \( 0 \neq x \in R \) such that \( x^p = 0 \)), then the Frobenius map can be thought of as the inclusion:

\[
R^p \subset R \text{ or the inclusion } R \subset R^{1/p}.
\]
Notation for Frobenius

- We want to explore the behavior of Frobenius on “nice rings”?
- We want to view $R$ as an $R$-module via the action of Frobenius.
- People often use $F_\ast R$ to denote the $R$-module which is equal to $R$ as an additive group, and where the $R$-module action is given by $r \cdot x = r^p x$.
  - One can also think of $F_\ast R$ as $R^{1/p}$. 
We want to explore the behavior of Frobenius on “nice rings”? 

We want to view $R$ as an $R$-module via the action of Frobenius.

People often use $F_\ast R$ to denote the $R$-module which is equal to $R$ as an additive group, and where the $R$-module action is given by $r \cdot x = r^p x$.

One can also think of $F_\ast R$ as $R^{1/p}$.
Notation for Frobenius

- We want to explore the behavior of Frobenius on “nice rings”?
- We want to view $R$ as an $R$-module via the action of Frobenius.
- People often use $F_*R$ to denote the $R$-module which is equal to $R$ as an additive group, and where the $R$-module action is given by $r.x = r^p x$.
  - One can also think of $F_*R$ as $R^{1/p}$.

Karl Schwede
Notation for Frobenius

- We want to explore the behavior of Frobenius on "nice rings"?
- We want to view $R$ as an $R$-module via the action of Frobenius.
- People often use $F_* R$ to denote the $R$-module which is equal to $R$ as an additive group, and where the $R$-module action is given by $r . x = r^p x$.
  - One can also think of $F_* R$ as $R^{1/p}$.
We want to explore the behavior of Frobenius on “nice rings”? 

We want to view $R$ as an $R$-module via the action of Frobenius.

People often use $F_* R$ to denote the $R$-module which is equal to $R$ as an additive group, and where the $R$-module action is given by $r.x = r^p x$.

One can also think of $F_* R$ as $R^{1/p}$. 
Smooth points and the action of Frobenius

Consider the ring $R = k[x]$ (polynomials in a single variable).

- If $k = \mathbb{C}$, then the ring $R$ would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It’s easy to see that $F_* R$ is free of rank $p$ (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

**Theorem (Kunz)**

A local domain $R$ of characteristic $p$ is regular (i.e., non-singular) if and only if $F_* R$ is flat as an $R$-module.

In our context, this implies that $R$ is smooth if and only if $F_* R$ is locally free.
Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring $R$ would correspond to the variety $\mathbb{C}$ (which is very smooth).
  - It’s easy to see that $F_* R$ is free of rank $p$ (with generators $1, x, \ldots, x^{p-1}$).
  - It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
  - In fact, there is the following theorem:

**Theorem (Kunz)**

A local domain $R$ of characteristic $p$ is regular (i.e., non-singular) if and only if $F_* R$ is flat as an $R$-module.

- In our context, this implies that $R$ is smooth if and only if $F_* R$ is locally free.
Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring $R$ would correspond to the variety $\mathbb{C}$ (which is very smooth).
  - It’s easy to see that $F_* R$ is free of rank $p$ (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

Theorem (Kunz)

A local domain $R$ of characteristic $p$ is regular (ie, non-singular) if and only if $F_* R$ is flat as an $R$-module.

- In our context, this implies that $R$ is smooth if and only if $F_* R$ is locally free.
Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring $R$ would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It’s easy to see that $F_* R$ is free of rank $p$ (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

**Theorem (Kunz)**

A local domain $R$ of characteristic $p$ is regular (ie, non-singular) if and only if $F_* R$ is flat as an $R$-module.

- In our context, this implies that $R$ is smooth if and only if $F_* R$ is locally free.
Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring $R$ would correspond to the variety $\mathbb{C}$ (which is very smooth).
- It’s easy to see that $F_*R$ is free of rank $p$ (with generators $1, x, \ldots, x^{p-1}$).
- It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
- In fact, there is the following theorem:

**Theorem (Kunz)**

A local domain $R$ of characteristic $p$ is regular (ie, non-singular) if and only if $F_*R$ is flat as an $R$-module.

- In our context, this implies that $R$ is smooth if and only if $F_*R$ is locally free.
Smooth points and the action of Frobenius

- Consider the ring $R = k[x]$ (polynomials in a single variable).
  - If $k = \mathbb{C}$, then the ring $R$ would correspond to the variety $\mathbb{C}$ (which is very smooth).
  - It’s easy to see that $F_\ast R$ is free of rank $p$ (with generators $1, x, \ldots, x^{p-1}$).
  - It turns out that any polynomial ring is free when viewed as a module via the action of Frobenius.
  - In fact, there is the following theorem:

**Theorem (Kunz)**

A local domain $R$ of characteristic $p$ is regular (ie, non-singular) if and only if $F_\ast R$ is flat as an $R$-module.

- In our context, this implies that $R$ is smooth if and only if $F_\ast R$ is locally free.
How can we use Frobenius to classify singularities?

**Definition**

A ring $R$ is said to be F-*pure* (or $F$-split) if there exists a surjective map of $R$-modules $\phi : F_* R \to R$.

- If $F_* R$ is free as an $R$-module, it is not hard to see that the Frobenius map splits.
Singularities on algebraic varieties
Types of singularities in characteristic zero
Singularities in characteristic $p > 0$

Definitions
Characteristic 0 vs characteristic $p > 0$ singularities

Singularities defined by Frobenius

- How can we use Frobenius to classify singularities?

**Definition**

A ring $R$ is said to be *F-pure* (or *F*-split) if there exists a surjective map of $R$-modules $\phi : F_* R \rightarrow R$.

- If $F_* R$ is free as an $R$-module, it is not hard to see that the Frobenius map splits.
An aside on Frobenius split varieties

- If $X$ is smooth and projective one can still restrict to open sets, corresponding to rings $R$, which are Frobenius split.
- Therefore, every smooth variety is “locally” Frobenius split.
- However, the various splittings $\phi$ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.
An aside on Frobenius split varieties

- If $X$ is smooth and projective one can still restrict to open sets, corresponding to rings $R$, which are Frobenius split.
- Therefore, every smooth variety is “locally” Frobenius split.
- However, the various splittings $\phi$ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.
An aside on Frobenius split varieties

- If $X$ is smooth and projective one can still restrict to open sets, corresponding to rings $R$, which are Frobenius split.
- Therefore, every smooth variety is “locally” Frobenius split.
- However, the the various splittings $\phi$ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.
An aside on Frobenius split varieties

- If $X$ is smooth and projective one can still restrict to open sets, corresponding to rings $R$, which are Frobenius split.
- Therefore, every smooth variety is “locally” Frobenius split.
- However, the the various splittings $\phi$ are often not compatible.
- Being globally Frobenius split is much more restrictive than being locally Frobenius split.
More singularities defined by Frobenius

- Other closely related classes of rings include:
  - (strongly) $F$-regular, $F$-injective, and $F$-rational.

![Diagram]

- Test ideals (from tight closure theory), $F$-pure thresholds, and $F$-pure centers also fit into this framework.
More singularities defined by Frobenius

- Other closely related classes of rings include:
  - (strongly) \( F \)-regular, \( F \)-injective, and \( F \)-rational.

Test ideals (from tight closure theory), \( F \)-pure thresholds, and \( F \)-pure centers also fit into this framework.
More singularities defined by Frobenius

- Other closely related classes of rings include:
  - (strongly) $F$-regular, $F$-injective, and $F$-rational.

$F$-regular $\leftrightarrow$ $F$-rational
\[\downarrow\downarrow\]
$F$-pure $\quad$ $\rightarrow$ $\quad$ $F$-injective
\[\searrow\nearrow\]
$F$-pure thresholds, and $F$-pure centers also fit into this framework.
Examples of singularities defined by Frobenius

- The ring corresponding to the quadric cone $R = k[x, y, z]/(x^2 + y^2 - z^2)$ is $F$-regular (except in characteristic 2).

- The ring corresponding to the cubic cone $R = k[x, y, z]/(x^3 + y^3 + z^3)$ is $F$-pure if and only if $p \equiv 1 \mod 3$. It is never $F$-rational.

- The ring corresponding to the node is $F$-pure but not $F$-rational.
Examples of singularities defined by Frobenius

- The ring corresponding to the quadric cone
  \( R = k[x, y, z]/(x^2 + y^2 - z^2) \) is \( F \)-regular (except in characteristic 2).

- The ring corresponding to the cubic cone
  \( R = k[x, y, z]/(x^3 + y^3 + z^3) \) is \( F \)-pure if and only if \( p \equiv 1 \mod 3 \). It is never \( F \)-rational.

- The ring corresponding to the node is \( F \)-pure but not \( F \)-rational.
Examples of singularities defined by Frobenius

- The ring corresponding to the quadric cone $R = k[x, y, z]/(x^2 + y^2 - z^2)$ is $F$-regular (except in characteristic 2).

- The ring corresponding to the cubic cone $R = k[x, y, z]/(x^3 + y^3 + z^3)$ is $F$-pure if and only if $p \equiv 1 \mod 3$. It is never $F$-rational.

- The ring corresponding to the node is $F$-pure but not $F$-rational.
Reduction to characteristic $p > 0$

- Suppose you have a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$.
- If the coefficients of $f$ are integers, then one can also view this as a variety in characteristic $p > 0$.
  - Squint hard, and study the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$
- One says that $X$ has *dense $F$-pure type* if for infinitely many $p$, the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ is $F$-pure.
  - One can similarly define *dense $F$-regular type*, etc.
- If the coefficients of $f$ are not integers, one can do something similar.
Suppose you have a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$.

If the coefficients of $f$ are integers, then one can also view this as a variety in characteristic $p > 0$.

- Squint hard, and study the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$

One says that $X$ has dense $F$-pure type if for infinitely many $p$, the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ is $F$-pure.

- One can similarly define dense $F$-regular type, etc.

If the coefficients of $f$ are not integers, one can do something similar.
Suppose you have a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$.

If the coefficients of $f$ are integers, then one can also view this as a variety in characteristic $p > 0$.

- Squint hard, and study the ring $\overline{\mathbb{F}}_p[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$

One says that $X$ has dense $F$-pure type if for infinitely many $p$, the ring $\overline{\mathbb{F}}_p[x_1, \ldots, x_n]/(f)$ is $F$-pure.

- One can similarly define dense $F$-regular type, etc.

If the coefficients of $f$ are not integers, one can do something similar.
Suppose you have a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$.

If the coefficients of $f$ are integers, then one can also view this as a variety in characteristic $p > 0$.

- Squint hard, and study the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$.

One says that $X$ has **dense $F$-pure type** if for infinitely many $p$, the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ is $F$-pure.

- One can similarly define **dense $F$-regular type**, etc.

If the coefficients of $f$ are not integers, one can do something similar.
Suppose you have a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$.

- If the coefficients of $f$ are integers, then one can also view this as a variety in characteristic $p > 0$.
  - Squint hard, and study the ring $\overline{F}_p[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$

One says that $X$ has **dense $F$-pure type** if for infinitely many $p$, the ring $\overline{F}_p[x_1, \ldots, x_n]/(f)$ is $F$-pure.

- One can similarly define **dense $F$-regular type**, etc.

- If the coefficients of $f$ are not integers, one can do something similar.
Suppose you have a complex affine variety $X$ defined by an equation $f(x_1, \ldots, x_n) = 0$.

If the coefficients of $f$ are integers, then one can also view this as a variety in characteristic $p > 0$.

- Squint hard, and study the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ instead of the ring $\mathbb{C}[x_1, \ldots, x_n]/(f)$

One says that $X$ has dense $F$-pure type if for infinitely many $p$, the ring $\mathbb{F}_p[x_1, \ldots, x_n]/(f)$ is $F$-pure.

- One can similarly define dense $F$-regular type, etc.

If the coefficients of $f$ are not integers, one can do something similar.
Since about 1980, people have been aware of connections between singularities defined by the action of Frobenius and singularities defined by a resolution of singularities.

Although the various classes of singularities were introduced independently.

After the introduction of tight closure by Hochster and Huneke, people began to make the correspondence precise. For example,

**Theorem (Smith, Hara/ Mehta-Srinivas)**

\[ X \text{ has rational singularities if and only if } X \text{ has dense F-rational type. } \]
Relation between the singularities

- Since about 1980, people have been aware of connections between singularities defined by the action of Frobenius and singularities defined by a resolution of singularities.
- Although the various classes of singularities were introduced independently.
- After the introduction of tight closure by Hochster and Huneke, people began to make the correspondence precise. For example,

**Theorem (Smith, Hara/ Mehta-Srinivas)**

\[ X \text{ has rational singularities if and only if } X \text{ has dense } F\text{-rational type.} \]
Relation between the singularities

Since about 1980, people have been aware of connections between singularities defined by the action of Frobenius and singularities defined by a resolution of singularities.

Although the various classes of singularities were introduced independently.

After the introduction of tight closure by Hochster and Huneke, people began to make the correspondence precise. For example,

Theorem (Smith, Hara/ Mehta-Srinivas)

\( X \) has rational singularities if and only if \( X \) has dense F-rational type.
More relations between the singularities

Many other people have since contributed to this dictionary: Fedder, Hara, Mehta, Mustață, –, Smith, Srinivas, Takagi, Watanabe, Yoshida and others.

**Theorem (–)**

If $X$ has dense $F$-injective type then $X$ has Du Bois singularities.

**Theorem (–)**

If $W \subseteq X$ is a log canonical center, then after reduction to characteristic $p \gg 0$, $W_p \subseteq X_p$ is a $F$-pure center.

Centers of $F$-purity are very closely related to compatibly Frobenius split subvarieties (which show up often in representation theory).
More relations between the singularities

Many other people have since contributed to this dictionary: Fedder, Hara, Mehta, Mustaţă, –, Smith, Srinivas, Takagi, Watanabe, Yoshida and others.

Theorem (–)

If \( X \) has dense \( F \)-injective type then \( X \) has Du Bois singularities.

Theorem (–)

If \( W \subseteq X \) is a log canonical center, then after reduction to characteristic \( p \gg 0 \), \( W_p \subseteq X_p \) is a \( F \)-pure center.

Centers of \( F \)-purity are very closely related to compatibly Frobenius split subvarieties (which show up often in representation theory).
More relations between the singularities

- Many other people have since contributed to this dictionary: Fedder, Hara, Mehta, Mustață, –, Smith, Srinivas, Takagi, Watanabe, Yoshida and others.

**Theorem (–)**

*If X has dense F-injective type then X has Du Bois singularities.*

**Theorem (–)**

*If W ⊆ X is a log canonical center, then after reduction to characteristic p ≫ 0, W_p ⊆ X_p is a F-pure center.*

- Centers of F-purity are very closely related to compatibly Frobenius split subvarieties (which show up often in representation theory).
The diagram

Terminal \[\downarrow\] Canonical
\[\downarrow\] Log Terminal \[\longrightarrow\] Rational
\[\downarrow\] Log Canonical \[\longrightarrow\] Du Bois
\[\longrightarrow\] F-Regular \[\longrightarrow\] F-Rational
\[\longrightarrow\] F-Pure \[\longrightarrow\] F-Injective

Multiplier ideals \[\longleftrightarrow\] Test ideals

LC Centers \[\longrightarrow\] F-pure centers

Definitions
Characteristic 0 vs characteristic \(p > 0\) singularities

Singularities on algebraic varieties
Types of singularities in characteristic zero
Singularities in characteristic \(p > 0\)
Remarks on the diagram

- It is unknown whether there are $F$-analogues of canonical or terminal singularities.
- It is conjectured that log canonical singularities are of dense $F$-pure type, but this is (very) open.
- Of course, this diagram has been used to inspire questions in both contexts. It has also been used to answer questions.
Remarks on the diagram

- It is unknown whether there are $F$-analogues of canonical or terminal singularities.
- It is conjectured that log canonical singularities are of dense $F$-pure type, but this is (very) open.
- Of course, this diagram has been used to inspire questions in both contexts. It has also been used to answer questions.
Remarks on the diagram

- It is unknown whether there are $F$-analogues of canonical or terminal singularities.
- It is conjectured that log canonical singularities are of dense $F$-pure type, but this is (very) open.
- Of course, this diagram has been used to inspire questions in both contexts. It has also been used to answer questions.
Suppose $X$ is an affine variety and $\alpha$ is an ideal (on the corresponding ring).

One then can define the multiplier ideal $\mathcal{J}(X, \alpha^t)$ where $t > 0$ is a real number.

As one increases $t$, these become smaller ideals.

$\mathcal{J}(X, \alpha^{t_1}) \supseteq \mathcal{J}(X, \alpha^{t_2}) \supseteq \mathcal{J}(X, \alpha^{t_3}) \supseteq \ldots$

They change at a discrete set of rational numbers $t_i$, called jumping numbers.

At least when $X$ is normal and $\mathbb{Q}$-Gorenstein.
Multiplier ideals

- Suppose $X$ is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases $t$, these become smaller ideals.

$$\mathcal{J}(X, \mathfrak{a}^{t_1}) \supseteq \mathcal{J}(X, \mathfrak{a}^{t_2}) \supseteq \mathcal{J}(X, \mathfrak{a}^{t_3}) \supseteq \ldots$$

- They change at a discrete set of rational numbers $t_i$, called *jumping numbers*.
  - At least when $X$ is normal and $\mathbb{Q}$-Gorenstein.
Suppose $X$ is an affine variety and $\alpha$ is an ideal (on the corresponding ring).

One then can define the multiplier ideal $\mathcal{I}(X, \alpha^t)$ where $t > 0$ is a real number.

As one increases $t$, these become smaller ideals.

$$\mathcal{I}(X, \alpha^{t_1}) \supseteq \mathcal{I}(X, \alpha^{t_2}) \supseteq \mathcal{I}(X, \alpha^{t_3}) \supseteq \ldots$$

They change at a discrete set of rational numbers $t_i$, called *jumping numbers*.

At least when $X$ is normal and $\mathbb{Q}$-Gorenstein.
Multiplier ideals

- Suppose $X$ is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{J}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases $t$, these become smaller ideals.

$$\mathcal{J}(X, \mathfrak{a}^{t_1}) \supseteq \mathcal{J}(X, \mathfrak{a}^{t_2}) \supseteq \mathcal{J}(X, \mathfrak{a}^{t_3}) \supseteq \ldots$$

- They change at a discrete set of rational numbers $t_i$, called jumping numbers.
  - At least when $X$ is normal and $\mathbb{Q}$-Gorenstein.
Multiplier ideals

- Suppose $X$ is an affine variety and $\mathfrak{a}$ is an ideal (on the corresponding ring).
- One then can define the multiplier ideal $\mathcal{I}(X, \mathfrak{a}^t)$ where $t > 0$ is a real number.
- As one increases $t$, these become smaller ideals.

$$\mathcal{I}(X, \mathfrak{a}^{t_1}) \supseteq \mathcal{I}(X, \mathfrak{a}^{t_2}) \supseteq \mathcal{I}(X, \mathfrak{a}^{t_3}) \supseteq \ldots$$

- They change at a discrete set of rational numbers $t_i$, called *jumping numbers*.
- At least when $X$ is normal and $\mathbb{Q}$-Gorenstein.
Frobenius jumping numbers

- But the test ideal, $\tau(X, a^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same “jumping” behavior holds (for a fixed $a$).

Theorem (Blickle, Mustață, Smith)

The set of “F-jumping numbers” for $a$ are discrete and rational when $X$ is smooth.

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

Theorem (–, Takagi, Zhang)

The set of “F-jumping numbers” for $a$ are discrete and rational when $X$ is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p$. 
**Frobenius jumping numbers**

- But the test ideal, $\tau(X, \alpha^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same "jumping" behavior holds (for a fixed $\alpha$).

**Theorem (Blickle, Mustaţă, Smith)**

*The set of “F-jumping numbers” for $\alpha$ are discrete and rational when $X$ is smooth.*

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

**Theorem (–, Takagi, Zhang)**

*The set of “F-jumping numbers” for $\alpha$ are discrete and rational when $X$ is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p$.*
Frobenius jumping numbers

- But the test ideal, $\tau(X, \alpha^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same "jumping" behavior holds (for a fixed $\alpha$).

**Theorem (Blickle, Mustață, Smith)**

*The set of “F-jumping numbers” for $\alpha$ are discrete and rational when $X$ is smooth.*

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

**Theorem (–, Takagi, Zhang)**

*The set of “F-jumping numbers” for $\alpha$ are discrete and rational when $X$ is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p$. 
Frobenius jumping numbers

- But the test ideal, $\tau(X, \alpha^t)$ is an analogue of the multiplier ideal.
- One can ask whether the same “jumping” behavior holds (for a fixed $\alpha$).

**Theorem (Blickle, Mustaţă, Smith)**

*The set of “F-jumping numbers” for $\alpha$ are discrete and rational when $X$ is smooth.*

- Also see [Monsky, Hara] and [Katzman, Lyubeznik, Zhang].

**Theorem (–, Takagi, Zhang)**

*The set of “F-jumping numbers” for $\alpha$ are discrete and rational when $X$ is normal and $\mathbb{Q}$-Gorenstein with index not divisible by $p$.**