Discreteness and rationality of F-jumping numbers on rings with singularities

Karl Schwede[1], Wenliang Zhang[1], Shunsuke Takagi[2]

[1]Department of Mathematics
University of Michigan
[2]Department of Mathematics
Kyushu University

Sectional AMS Meeting – Spring 2009

Outline

Outline

Outline

Multiplier ideals on singular varieties

- Suppose that $X = \operatorname{Spec} R$ is normal (of finite type $/\mathbb{C}$).
- We let Δ be an effective $\mathbb{Q}$-divisor.
  - A *$\mathbb{Q}$-divisor* is a linear combination of subvarieties of codimension 1 such with positive rational coefficients.
- We also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Here K_X is a divisor in the whose divisor class corresponds to ω_X.
  - *$\mathbb{Q}$-Cartier* means that there exists some $n \in \mathbb{Z}$ such that $n\Delta$ is integral (all denominators were cleared) and $nK_X + n\Delta$ is Cartier (ie, locally trivial in the divisor class group).
  - When X is $\mathbb{Q}$-Gorenstein (that means nK_X is Cartier, locally $\omega_X^{(n)} \cong R$, for some n), we can choose $\Delta = 0$.
- Then for any ideal $\mathfrak{a}$ on X, the setting of a triple $(X, \Delta, \mathfrak{a}^t)$ (for $t \in \mathbb{R}_{\geq 0}$) is the natural context for considering multiplier ideals from the point of view of the "MMP".

Multiplier ideals on singular varieties

- Suppose that $X = \operatorname{Spec} R$ is normal (of finite type $/\mathbb{C}$).
- We let Δ be an effective $\mathbb{Q}$-divisor.
  - A *$\mathbb{Q}$-divisor* is a linear combination of subvarieties of codimension 1 such with positive rational coefficients.
- We also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Here K_X is a divisor in the whose divisor class corresponds to ω_X.
  - *$\mathbb{Q}$-Cartier* means that there exists some $n \in \mathbb{Z}$ such that $n\Delta$ is integral (all denominators were cleared) and $nK_X + n\Delta$ is Cartier (ie, locally trivial in the divisor class group).
  - When X is $\mathbb{Q}$-Gorenstein (that means nK_X is Cartier, locally $\omega_X^{(n)} \cong R$, for some n), we can choose $\Delta = 0$.
- Then for any ideal $\mathfrak{a}$ on X, the setting of a triple $(X, \Delta, \mathfrak{a}^t)$ (for $t \in \mathbb{R}_{\geq 0}$) is the natural context for considering multiplier ideals from the point of view of the "MMP".

Multiplier ideals on singular varieties

- Suppose that $X = \operatorname{Spec} R$ is normal (of finite type $/\mathbb{C}$).
- We let Δ be an effective $\mathbb{Q}$-divisor.
  - A *$\mathbb{Q}$-divisor* is a linear combination of subvarieties of codimension 1 such with positive rational coefficients.
- We also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Here K_X is a divisor in the whose divisor class corresponds to ω_X.
  - *$\mathbb{Q}$-Cartier* means that there exists some $n \in \mathbb{Z}$ such that $n\Delta$ is integral (all denominators were cleared) and $nK_X + n\Delta$ is Cartier (ie, locally trivial in the divisor class group).
  - When X is $\mathbb{Q}$-Gorenstein (that means nK_X is Cartier, locally $\omega_X^{(n)} \cong R$, for some n), we can choose $\Delta = 0$.
- Then for any ideal $\mathfrak{a}$ on X, the setting of a triple $(X, \Delta, \mathfrak{a}^t)$ (for $t \in \mathbb{R}_{\geq 0}$) is the natural context for considering multiplier ideals from the point of view of the "MMP".

Multiplier ideals on singular varieties

- Suppose that $X = \operatorname{Spec} R$ is normal (of finite type $/\mathbb{C}$).
- We let Δ be an effective $\mathbb{Q}$-divisor.
  - A *$\mathbb{Q}$-divisor* is a linear combination of subvarieties of codimension 1 such with positive rational coefficients.
- We also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Here K_X is a divisor in the whose divisor class corresponds to ω_X.
  - *$\mathbb{Q}$-Cartier* means that there exists some $n \in \mathbb{Z}$ such that $n\Delta$ is integral (all denominators were cleared) and $nK_X + n\Delta$ is Cartier (ie, locally trivial in the divisor class group).
  - When X is $\mathbb{Q}$-Gorenstein (that means nK_X is Cartier, locally $\omega_X^{(n)} \cong R$, for some n), we can choose $\Delta = 0$.
- Then for any ideal $\mathfrak{a}$ on X, the setting of a triple $(X, \Delta, \mathfrak{a}^t)$ (for $t \in \mathbb{R}_{\geq 0}$) is the natural context for considering multiplier ideals from the point of view of the "MMP".

Multiplier ideals on singular varieties

- Suppose that $X = \operatorname{Spec} R$ is normal (of finite type $/\mathbb{C}$).
- We let Δ be an effective $\mathbb{Q}$-divisor.
  - A *$\mathbb{Q}$-divisor* is a linear combination of subvarieties of codimension 1 such with positive rational coefficients.
- We also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Here K_X is a divisor in the whose divisor class corresponds to ω_X.
  - *$\mathbb{Q}$-Cartier* means that there exists some $n \in \mathbb{Z}$ such that $n\Delta$ is integral (all denominators were cleared) and $nK_X + n\Delta$ is Cartier (ie, locally trivial in the divisor class group).
  - When X is $\mathbb{Q}$-Gorenstein (that means nK_X is Cartier, locally $\omega_X^{(n)} \cong R$, for some n), we can choose $\Delta = 0$.
- Then for any ideal $\mathfrak{a}$ on X, the setting of a triple $(X, \Delta, \mathfrak{a}^t)$ (for $t \in \mathbb{R}_{\geq 0}$) is the natural context for considering multiplier ideals from the point of view of the "MMP".

Multiplier ideals on singular varieties

- Suppose that $X = \operatorname{Spec} R$ is normal (of finite type $/\mathbb{C}$).
- We let Δ be an effective $\mathbb{Q}$-divisor.
  - A *$\mathbb{Q}$-divisor* is a linear combination of subvarieties of codimension 1 such with positive rational coefficients.
- We also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Here K_X is a divisor in the whose divisor class corresponds to ω_X.
  - *$\mathbb{Q}$-Cartier* means that there exists some $n \in \mathbb{Z}$ such that $n\Delta$ is integral (all denominators were cleared) and $nK_X + n\Delta$ is Cartier (ie, locally trivial in the divisor class group).
  - When X is $\mathbb{Q}$-Gorenstein (that means nK_X is Cartier, locally $\omega_X^{(n)} \cong R$, for some n), we can choose $\Delta = 0$.
- Then for any ideal $\mathfrak{a}$ on X, the setting of a triple $(X, \Delta, \mathfrak{a}^t)$ (for $t \in \mathbb{R}_{\geq 0}$) is the natural context for considering multiplier ideals from the point of view of the "MMP".

Multiplier ideals on singular varieties

- Suppose that $X = \operatorname{Spec} R$ is normal (of finite type $/\mathbb{C}$).
- We let Δ be an effective $\mathbb{Q}$-divisor.
  - A *$\mathbb{Q}$-divisor* is a linear combination of subvarieties of codimension 1 such with positive rational coefficients.
- We also assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. Here K_X is a divisor in the whose divisor class corresponds to ω_X.
  - *$\mathbb{Q}$-Cartier* means that there exists some $n \in \mathbb{Z}$ such that $n\Delta$ is integral (all denominators were cleared) and $nK_X + n\Delta$ is Cartier (ie, locally trivial in the divisor class group).
  - When X is $\mathbb{Q}$-Gorenstein (that means nK_X is Cartier, locally $\omega_X^{(n)} \cong R$, for some n), we can choose $\Delta = 0$.
- Then for any ideal $\mathfrak{a}$ on X, the setting of a triple $(X, \Delta, \mathfrak{a}^t)$ (for $t \in \mathbb{R}_{\geq 0}$) is the natural context for considering multiplier ideals from the point of view of the "MMP".

The definition of multiplier ideals

- Take a log resolution $\pi : \widetilde{X} \to X$ with $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$.
  - I'm not going to give a precise definition here.
- Then (using this $\mathbb{Q}$-Cartier notion), we can define the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ to be

$$\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil).$$

  - The round-up just rounds up the coefficients of the $\mathbb{Q}$-divisors.
- Another way to think of this is that there are a finite number of discrete valuations v_i (of Frac R) and integers m_i and $n_i > 0$ such that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \{ r \in R \mid v_i(r) \geq \lfloor n_i t + m_i \rfloor \}$$

- Here n_i is just the order of $\mathfrak{a}$ along v_i and the m_i depend on Δ, v_i and the singularities of X.

The definition of multiplier ideals

- Take a log resolution $\pi : \widetilde{X} \to X$ with $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$.
  - I'm not going to give a precise definition here.
- Then (using this $\mathbb{Q}$-Cartier notion), we can define the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ to be

$$\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil).$$

  - The round-up just rounds up the coefficients of the $\mathbb{Q}$-divisors.
- Another way to think of this is that there are a finite number of discrete valuations v_i (of Frac R) and integers m_i and $n_i > 0$ such that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor\}$$

- Here n_i is just the order of $\mathfrak{a}$ along v_i and the m_i depend on Δ, v_i and the singularities of X.

The definition of multiplier ideals

- Take a log resolution $\pi : \widetilde{X} \to X$ with $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$.
  - I'm not going to give a precise definition here.
- Then (using this $\mathbb{Q}$-Cartier notion), we can define the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ to be

$$\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil).$$

  - The round-up just rounds up the coefficients of the $\mathbb{Q}$-divisors.
- Another way to think of this is that there are a finite number of discrete valuations v_i (of Frac R) and integers m_i and $n_i > 0$ such that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \{ r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor \}$$

- Here n_i is just the order of $\mathfrak{a}$ along v_i and the m_i depend on Δ, v_i and the singularities of X.

The definition of multiplier ideals

- Take a log resolution $\pi : \widetilde{X} \to X$ with $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$.
  - I'm not going to give a precise definition here.
- Then (using this $\mathbb{Q}$-Cartier notion), we can define the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ to be

$$\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil).$$

  - The round-up just rounds up the coefficients of the $\mathbb{Q}$-divisors.
- Another way to think of this is that there are a finite number of discrete valuations v_i (of Frac R) and integers m_i and $n_i > 0$ such that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor\}$$

- Here n_i is just the order of $\mathfrak{a}$ along v_i and the m_i depend on Δ, v_i and the singularities of X.

The definition of multiplier ideals

- Take a log resolution $\pi : \widetilde{X} \to X$ with $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$.
  - I'm not going to give a precise definition here.
- Then (using this $\mathbb{Q}$-Cartier notion), we can define the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ to be

$$\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil).$$

  - The round-up just rounds up the coefficients of the $\mathbb{Q}$-divisors.
- Another way to think of this is that there are a finite number of discrete valuations v_i (of Frac R) and integers m_i and $n_i > 0$ such that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor\}$$

- Here n_i is just the order of $\mathfrak{a}$ along v_i and the m_i depend on Δ, v_i and the singularities of X.

The definition of multiplier ideals

- Take a log resolution $\pi : \widetilde{X} \to X$ with $\mathfrak{a}\mathcal{O}_{\widetilde{X}} = \mathcal{O}_{\widetilde{X}}(-E)$.
    - I'm not going to give a precise definition here.
- Then (using this $\mathbb{Q}$-Cartier notion), we can define the multiplier ideal $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ to be

$$\pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil).$$

    - The round-up just rounds up the coefficients of the $\mathbb{Q}$-divisors.
- Another way to think of this is that there are a finite number of discrete valuations v_i (of Frac R) and integers m_i and $n_i > 0$ such that

$$\mathcal{J}(X, \Delta, \mathfrak{a}^t) = \{ r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor \}$$

- Here n_i is just the order of $\mathfrak{a}$ along v_i and the m_i depend on Δ, v_i and the singularities of X.

Jumping numbers

- So consider a X, Δ, and $\mathfrak{a}^t$ as before.
- And consider what happens to the multiplier ideals $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ as one varies t.
- That is, consider what happens to $\mathcal{J}(X, \Delta, \mathfrak{a}^t) =$

$$\{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor \} = \pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil)$$

  as one varies t (for a fixed log resolution $\pi : \widetilde{X} \to X$).
- Of course, because of the round up / down, this ideal only changes at a discrete set of rational numbers.
- These are called the *jumping numbers of* $(X, \Delta, \mathfrak{a}^t)$. They were introduced by Ein-Lazarsfeld-Smith-Varolin.

Jumping numbers

- So consider a X, Δ, and $\mathfrak{a}^t$ as before.
- And consider what happens to the multiplier ideals $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ as one varies t.
- That is, consider what happens to $\mathcal{J}(X, \Delta, \mathfrak{a}^t) =$

$$\{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor\} = \pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil)$$

  as one varies t (for a fixed log resolution $\pi : \widetilde{X} \to X$).
- Of course, because of the round up / down, this ideal only changes at a discrete set of rational numbers.
- These are called the *jumping numbers of* $(X, \Delta, \mathfrak{a}^t)$. They were introduced by Ein-Lazarsfeld-Smith-Varolin.

Jumping numbers

- So consider a X, Δ, and $\mathfrak{a}^t$ as before.
- And consider what happens to the multiplier ideals $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ as one varies t.
- That is, consider what happens to $\mathcal{J}(X, \Delta, \mathfrak{a}^t) =$

$$\{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor\} = \pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil)$$

as one varies t (for a fixed log resolution $\pi : \widetilde{X} \to X$).

- Of course, because of the round up / down, this ideal only changes at a discrete set of rational numbers.
- These are called the *jumping numbers of* $(X, \Delta, \mathfrak{a}^t)$. They were introduced by Ein-Lazarsfeld-Smith-Varolin.

Jumping numbers

- So consider a X, Δ, and $\mathfrak{a}^t$ as before.
- And consider what happens to the multiplier ideals $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ as one varies t.
- That is, consider what happens to $\mathcal{J}(X, \Delta, \mathfrak{a}^t) =$

  $$\{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor\} = \pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil)$$

  as one varies t (for a fixed log resolution $\pi : \widetilde{X} \to X$).
- Of course, because of the round up / down, this ideal only changes at a discrete set of rational numbers.
- These are called the *jumping numbers of* $(X, \Delta, \mathfrak{a}^t)$. They were introduced by Ein-Lazarsfeld-Smith-Varolin.

Jumping numbers

- So consider a X, Δ, and $\mathfrak{a}^t$ as before.
- And consider what happens to the multiplier ideals $\mathcal{J}(X, \Delta, \mathfrak{a}^t)$ as one varies t.
- That is, consider what happens to $\mathcal{J}(X, \Delta, \mathfrak{a}^t) =$

$$\{r \in R | v_i(r) \geq \lfloor n_i t + m_i \rfloor\} = \pi_* \mathcal{O}_{\widetilde{X}}(\lceil K_{\widetilde{X}} - \pi^*(K_X + \Delta) - tE \rceil)$$

  as one varies t (for a fixed log resolution $\pi : \widetilde{X} \to X$).
- Of course, because of the round up / down, this ideal only changes at a discrete set of rational numbers.
- These are called the *jumping numbers of* $(X, \Delta, \mathfrak{a}^t)$. They were introduced by Ein-Lazarsfeld-Smith-Varolin.

Examples and applications

- For example, if $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, $\Delta = 0$ and $\mathfrak{a} = (x^2, y^3)$. Then the jumping numbers are

$$\{5/6, 7/6, 11/6, 2, 13/6, 17/6, 3, \dots\}.$$

- The first jumping number is called the *log canonical threshold*. (In the above example, the log canonical threshold is $5/6$.)
- The study log canonical thresholds is an important part of the (MMP) minimal model program.
- In particular, one can explore the (still open) question "termination of flips" using log canonical thresholds. This is via Shokurov's ACC conjecture.

Examples and applications

- For example, if $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, $\Delta = 0$ and $\mathfrak{a} = (x^2, y^3)$. Then the jumping numbers are

$$\{5/6, 7/6, 11/6, 2, 13/6, 17/6, 3, \dots\}.$$

- The first jumping number is called the *log canonical threshold*. (In the above example, the log canonical threshold is $5/6$.)
- The study log canonical thresholds is an important part of the (MMP) minimal model program.
- In particular, one can explore the (still open) question "termination of flips" using log canonical thresholds. This is via Shokurov's ACC conjecture.

Examples and applications

- For example, if $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, $\Delta = 0$ and $\mathfrak{a} = (x^2, y^3)$. Then the jumping numbers are
$$\{5/6, 7/6, 11/6, 2, 13/6, 17/6, 3, \dots\}.$$
- The first jumping number is called the *log canonical threshold*. (In the above example, the log canonical threshold is $5/6$.)
- The study log canonical thresholds is an important part of the (MMP) minimal model program.
- In particular, one can explore the (still open) question "termination of flips" using log canonical thresholds. This is via Shokurov's ACC conjecture.

Examples and applications

- For example, if $X = \mathbb{A}^2 = \operatorname{Spec} k[x, y]$, $\Delta = 0$ and $\mathfrak{a} = (x^2, y^3)$. Then the jumping numbers are

$$\{5/6, 7/6, 11/6, 2, 13/6, 17/6, 3, \dots\}.$$

- The first jumping number is called the *log canonical threshold*. (In the above example, the log canonical threshold is $5/6$.)
- The study log canonical thresholds is an important part of the (MMP) minimal model program.
- In particular, one can explore the (still open) question "termination of flips" using log canonical thresholds. This is via Shokurov's ACC conjecture.

Outline

Generalized test ideals

- Hara and Yoshida introduced the notion of tight closure of pairs. So let $(R, \mathfrak{a}^t)$ be a pair of an F-finite domain R and an ideal $\mathfrak{a}$ such that $\mathfrak{a} \neq 0$.
- For any ideal $I = (x_1, \ldots, x_d) \subseteq R$ they define the tight closure of I, denoted $I^{*\mathfrak{a}^t}$ to be

$$\{x \in R | \exists c \in R \setminus \{0\}, c\mathfrak{a}^{\lceil t(p^e-1) \rceil} x^{p^e} \in I^{[p^e]} \; \forall \; e \geq 0\}$$

- An element $c \in R \setminus \{0\}$ is said to be a *sharp test element for* $(R, \mathfrak{a}^t)$ if $z \in I^{*\mathfrak{a}^t}$ implies that $c\mathfrak{a}^{\lceil t(p^e-1) \rceil} z^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. (This is a slight modification of the definition of Hara and Yoshida).
- The *test ideal of* $(R, \mathfrak{a}^t)$, denoted $\tau_R(\mathfrak{a}^t)$ is the ideal generated by all the sharp test elements of R.

Generalized test ideals

- Hara and Yoshida introduced the notion of tight closure of pairs. So let $(R, \mathfrak{a}^t)$ be a pair of an F-finite domain R and an ideal $\mathfrak{a}$ such that $\mathfrak{a} \neq 0$.
- For any ideal $I = (x_1, \dots, x_d) \subseteq R$ they define the tight closure of I, denoted $I^{*\mathfrak{a}^t}$ to be

$$\{x \in R | \exists c \in R \setminus \{0\}, c\mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e} \in I^{[p^e]} \ \forall\ e \geq 0\}$$

- An element $c \in R \setminus \{0\}$ is said to be a *sharp test element for* $(R, \mathfrak{a}^t)$ if $z \in I^{*\mathfrak{a}^t}$ implies that $c\mathfrak{a}^{\lceil t(p^e-1)\rceil} z^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. (This is a slight modification of the definition of Hara and Yoshida).
- The *test ideal of* $(R, \mathfrak{a}^t)$, denoted $\tau_R(\mathfrak{a}^t)$ is the ideal generated by all the sharp test elements of R.

Generalized test ideals

- Hara and Yoshida introduced the notion of tight closure of pairs. So let $(R, \mathfrak{a}^t)$ be a pair of an F-finite domain R and an ideal $\mathfrak{a}$ such that $\mathfrak{a} \neq 0$.
- For any ideal $I = (x_1, \ldots, x_d) \subseteq R$ they define the tight closure of I, denoted $I^{*\mathfrak{a}^t}$ to be

$$\{x \in R | \exists c \in R \setminus \{0\}, c\mathfrak{a}^{\lceil t(p^e-1) \rceil} x^{p^e} \in I^{[p^e]} \ \forall \ e \geq 0\}$$

- An element $c \in R \setminus \{0\}$ is said to be a *sharp test element for* $(R, \mathfrak{a}^t)$ if $z \in I^{*\mathfrak{a}^t}$ implies that $c\mathfrak{a}^{\lceil t(p^e-1) \rceil} z^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. (This is a slight modification of the definition of Hara and Yoshida).
- The *test ideal of* $(R, \mathfrak{a}^t)$, denoted $\tau_R(\mathfrak{a}^t)$ is the ideal generated by all the sharp test elements of R.

Generalized test ideals

- Hara and Yoshida introduced the notion of tight closure of pairs. So let $(R, \mathfrak{a}^t)$ be a pair of an F-finite domain R and an ideal $\mathfrak{a}$ such that $\mathfrak{a} \neq 0$.
- For any ideal $I = (x_1, \ldots, x_d) \subseteq R$ they define the tight closure of I, denoted $I^{*\mathfrak{a}^t}$ to be

$$\{x \in R | \exists c \in R \setminus \{0\}, c\mathfrak{a}^{\lceil t(p^e-1) \rceil} x^{p^e} \in I^{[p^e]} \ \forall \ e \geq 0\}$$

- An element $c \in R \setminus \{0\}$ is said to be a *sharp test element for* $(R, \mathfrak{a}^t)$ if $z \in I^{*\mathfrak{a}^t}$ implies that $c\mathfrak{a}^{\lceil t(p^e-1) \rceil} z^{p^e} \in I^{[p^e]}$ for all $e \geq 0$. (This is a slight modification of the definition of Hara and Yoshida).
- The *test ideal of* $(R, \mathfrak{a}^t)$, denoted $\tau_R(\mathfrak{a}^t)$ is the ideal generated by all the sharp test elements of R.

More on generalized test ideals

- If $(R, \mathfrak{a}^t)$ in characteristic $p \gg 0$ is reduced generically from a characteristic zero normal $\mathbb{Q}$-Gorenstein ring R_0 with ideal $\mathfrak{a}_0$, then $\tau_R(\mathfrak{a}^t)$ coincides with the reduction of the multiplier ideal $\mathcal{J}(\operatorname{Spec} R_0, \mathfrak{a}_0^t)$. [Hara, Yoshida]
  - However, the side of $p \gg 0$ needed depends on t.
- As t increases, one can show that $\tau_R(\mathfrak{a}^t)$ becomes smaller (but it's not clear if it jumps at a discrete set of rational numbers).
- Define an *F-jumping number of* $(R, \mathfrak{a}^t)$ to be a $t > 0$ such that $\tau_R(\mathfrak{a}^{t-\epsilon}) \neq \tau_R(\mathfrak{a}^t)$ for all sufficiently small $\epsilon > 0$.

More on generalized test ideals

- If $(R, \mathfrak{a}^t)$ in characteristic $p \gg 0$ is reduced generically from a characteristic zero normal $\mathbb{Q}$-Gorenstein ring R_0 with ideal $\mathfrak{a}_0$, then $\tau_R(\mathfrak{a}^t)$ coincides with the reduction of the multiplier ideal $\mathcal{J}(\operatorname{Spec} R_0, \mathfrak{a}_0^t)$. [Hara, Yoshida]
  - However, the side of $p \gg 0$ needed depends on t.
- As t increases, one can show that $\tau_R(\mathfrak{a}^t)$ becomes smaller (but it's not clear if it jumps at a discrete set of rational numbers).
- Define an *F-jumping number of $(R, \mathfrak{a}^t)$* to be a $t > 0$ such that $\tau_R(\mathfrak{a}^{t-\epsilon}) \neq \tau_R(\mathfrak{a}^t)$ for all sufficiently small $\epsilon > 0$.

More on generalized test ideals

- If $(R, \mathfrak{a}^t)$ in characteristic $p \gg 0$ is reduced generically from a characteristic zero normal $\mathbb{Q}$-Gorenstein ring R_0 with ideal $\mathfrak{a}_0$, then $\tau_R(\mathfrak{a}^t)$ coincides with the reduction of the multiplier ideal $\mathcal{J}(\operatorname{Spec} R_0, \mathfrak{a}_0^t)$. [Hara, Yoshida]
  - However, the side of $p \gg 0$ needed depends on t.
- As t increases, one can show that $\tau_R(\mathfrak{a}^t)$ becomes smaller (but it's not clear if it jumps at a discrete set of rational numbers).
- Define an *F-jumping number of* $(R, \mathfrak{a}^t)$ to be a $t > 0$ such that $\tau_R(\mathfrak{a}^{t-\epsilon}) \neq \tau_R(\mathfrak{a}^t)$ for all sufficiently small $\epsilon > 0$.

More on generalized test ideals

- If $(R, \mathfrak{a}^t)$ in characteristic $p \gg 0$ is reduced generically from a characteristic zero normal $\mathbb{Q}$-Gorenstein ring R_0 with ideal $\mathfrak{a}_0$, then $\tau_R(\mathfrak{a}^t)$ coincides with the reduction of the multiplier ideal $\mathcal{J}(\operatorname{Spec} R_0, \mathfrak{a}_0^t)$. [Hara, Yoshida]
  - However, the side of $p \gg 0$ needed depends on t.
- As t increases, one can show that $\tau_R(\mathfrak{a}^t)$ becomes smaller (but it's not clear if it jumps at a discrete set of rational numbers).
- Define an *F-jumping number of $(R, \mathfrak{a}^t)$* to be a $t > 0$ such that $\tau_R(\mathfrak{a}^{t-\epsilon}) \neq \tau_R(\mathfrak{a}^t)$ for all sufficiently small $\epsilon > 0$.

The question

- So it is natural to ask, are the set of F-jumping numbers a discrete set of rational numbers?
- Yes!
    - For R regular and finite type over a perfect field [Blickle, Mustaţă, Smith].
    - For R local regular and $\mathfrak{a}$ principal [Katzman, Lyubeznik, Zhang].
    - Other special cases are due to [Hara-Monsky], [Takagi] (and also [S., Takagi])

The question

- So it is natural to ask, are the set of F-jumping numbers a discrete set of rational numbers?
- Yes!
  - For R regular and finite type over a perfect field [Blickle, Mustaţă, Smith].
  - For R local regular and $\mathfrak{a}$ principal [Katzman, Lyubeznik, Zhang].
  - Other special cases are due to [Hara-Monsky], [Takagi] (and also [S., Takagi])

The question

- So it is natural to ask, are the set of F-jumping numbers a discrete set of rational numbers?
- Yes!
  - For R regular and finite type over a perfect field [Blickle, Mustaţă, Smith].
  - For R local regular and $\mathfrak{a}$ principal [Katzman, Lyubeznik, Zhang].
  - Other special cases are due to [Hara-Monsky], [Takagi] (and also [S., Takagi])

The question

- So it is natural to ask, are the set of F-jumping numbers a discrete set of rational numbers?
- Yes!
  - For R regular and finite type over a perfect field [Blickle, Mustaţă, Smith].
  - For R local regular and $\mathfrak{a}$ principal [Katzman, Lyubeznik, Zhang].
  - Other special cases are due to [Hara-Monsky], [Takagi] (and also [S., Takagi])

The question

- So it is natural to ask, are the set of F-jumping numbers a discrete set of rational numbers?
- Yes!
  - For R regular and finite type over a perfect field [Blickle, Mustaţă, Smith].
  - For R local regular and $\mathfrak{a}$ principal [Katzman, Lyubeznik, Zhang].
  - Other special cases are due to [Hara-Monsky], [Takagi] (and also [S., Takagi])

$\mathbb{Q}$-divisors Δ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier

- Suppose that Δ is an effective $\mathbb{Q}$-divisor on Spec R (which is normal). One can define tight closure of an ideal I with respect to Δ (and you can throw in $\mathfrak{a}^t$ too). That is, you can define $I^{*\Delta, \mathfrak{a}^t}$.
- However, another way to think of it is (for a local ring), there is a bijection of sets

$$\left\{ \begin{array}{c} \text{Effective } \mathbb{Q}\text{-divisors } \Delta \\ \text{such that } (p^e - 1)(K_X + \Delta) \\ \text{is Cartier} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Nonzero elements of} \\ \mathrm{Hom}_R(R^{1/p^e}, R) \end{array} \right\} \Big/ \sim$$

- And if R is complete, then this is also equivalent to:

$$\left\{ \begin{array}{c} \text{Nonzero } R\{F^e\}\text{-module} \\ \text{structures on } E_R \end{array} \right\} \Big/ \sim$$

$\mathbb{Q}$-divisors Δ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier

- Suppose that Δ is an effective $\mathbb{Q}$-divisor on Spec R (which is normal). One can define tight closure of an ideal I with respect to Δ (and you can throw in $\mathfrak{a}^t$ too). That is, you can define $I^{*\Delta, \mathfrak{a}^t}$.
- However, another way to think of it is (for a local ring), there is a bijection of sets

$$\left\{ \begin{array}{c} \text{Effective } \mathbb{Q}\text{-divisors } \Delta \\ \text{such that } (p^e - 1)(K_X + \Delta) \\ \text{is Cartier} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Nonzero elements of} \\ \mathrm{Hom}_R(R^{1/p^e}, R) \end{array} \right\} \Big/ \sim$$

- And if R is complete, then this is also equivalent to:

$$\left\{ \begin{array}{c} \text{Nonzero } R\{F^e\}\text{-module} \\ \text{structures on } E_R \end{array} \right\} \Big/ \sim$$

$\mathbb{Q}$-divisors Δ such that $K_R + \Delta$ is $\mathbb{Q}$-Cartier

- Suppose that Δ is an effective $\mathbb{Q}$-divisor on $\operatorname{Spec} R$ (which is normal). One can define tight closure of an ideal I with respect to Δ (and you can throw in $\mathfrak{a}^t$ too). That is, you can define $I^{*\Delta,\mathfrak{a}^t}$.
- However, another way to think of it is (for a local ring), there is a bijection of sets

$$\left\{ \begin{array}{c} \text{Effective } \mathbb{Q}\text{-divisors } \Delta \\ \text{such that } (p^e - 1)(K_X + \Delta) \\ \text{is Cartier} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{Nonzero elements of} \\ \operatorname{Hom}_R(R^{1/p^e}, R) \end{array} \right\} \Big/ \sim$$

- And if R is complete, then this is also equivalent to:

$$\left\{ \begin{array}{c} \text{Nonzero } R\{F^e\}\text{-module} \\ \text{structures on } E_R \end{array} \right\} \Big/ \sim$$

Outline

The Katzman-Lyubeznik-Zhang argument

- One option is to modify the KLZ argument (that you just heard about). One can get the following theorem

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal local and $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of (R, Δ, f^t) is a discrete set of rational numbers.

- There are several places where you need to modify the original argument: (insert the test ideal $\tau(R, \Delta)$).
- In the the Katzman-Lyubeznik-Zhang argument, the real key point is the Hartshorne-Speiser-Lyubeznik theorem.

The Katzman-Lyubeznik-Zhang argument

- One option is to modify the KLZ argument (that you just heard about). One can get the following theorem

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal local and $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of (R, Δ, f^t) is a discrete set of rational numbers.

- There are several places where you need to modify the original argument: (insert the test ideal $\tau(R, \Delta)$).
- In the the Katzman-Lyubeznik-Zhang argument, the real key point is the Hartshorne-Speiser-Lyubeznik theorem.

The Katzman-Lyubeznik-Zhang argument

- One option is to modify the KLZ argument (that you just heard about). One can get the following theorem

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal local and $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of (R, Δ, f^t) is a discrete set of rational numbers.

- There are several places where you need to modify the original argument: (insert the test ideal $\tau(R, \Delta)$).
- In the the Katzman-Lyubeznik-Zhang argument, the real key point is the Hartshorne-Speiser-Lyubeznik theorem.

The Katzman-Lyubeznik-Zhang argument

- One option is to modify the KLZ argument (that you just heard about). One can get the following theorem

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal local and $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of (R, Δ, f^t) is a discrete set of rational numbers.

- There are several places where you need to modify the original argument: (insert the test ideal $\tau(R, \Delta)$).
- In the the Katzman-Lyubeznik-Zhang argument, the real key point is the Hartshorne-Speiser-Lyubeznik theorem.

Outside of the local setting?

- In the F-finite case, one can phrase a dual form of Hartshorne-Lyubeznik-Smith.

Question

Suppose that M is a finite R-module and that $\phi : M \to M$ is an additive map such that $\phi(r^{p^e} x) = r\phi(x)$. Let ϕ_n be the map obtained by composing ϕ with itself n-times. Does

$$\mathrm{Im}(\phi_1) \supseteq \mathrm{Im}(\phi_2) \supseteq \mathrm{Im}(\phi_3) \supseteq \dots \text{ stabilize?}$$

- If this is true, then one can modify the KLZ proof to work for any F-finite ring (not necessarily local).
- We can answer this question affirmatively for R of finite type over a perfect field.

Outside of the local setting?

- In the F-finite case, one can phrase a dual form of Hartshorne-Lyubeznik-Smith.

Question

Suppose that M is a finite R-module and that $\phi : M \to M$ is an additive map such that $\phi(r^{p^e} x) = r\phi(x)$. Let ϕ_n be the map obtained by composing ϕ with itself n-times. Does

$$\mathrm{Im}(\phi_1) \supseteq \mathrm{Im}(\phi_2) \supseteq \mathrm{Im}(\phi_3) \supseteq \dots \text{ stabilize?}$$

- If this is true, then one can modify the KLZ proof to work for any F-finite ring (not necessarily local).
- We can answer this question affirmatively for R of finite type over a perfect field.

Outside of the local setting?

- In the F-finite case, one can phrase a dual form of Hartshorne-Lyubeznik-Smith.

Question

Suppose that M is a finite R-module and that $\phi : M \to M$ is an additive map such that $\phi(r^{p^e} x) = r\phi(x)$. Let ϕ_n be the map obtained by composing ϕ with itself n-times. Does

$$\mathrm{Im}(\phi_1) \supseteq \mathrm{Im}(\phi_2) \supseteq \mathrm{Im}(\phi_3) \supseteq \dots \text{ stabilize?}$$

- If this is true, then one can modify the KLZ proof to work for any F-finite ring (not necessarily local).
- We can answer this question affirmatively for R of finite type over a perfect field.

Outside of the local setting?

- In the F-finite case, one can phrase a dual form of Hartshorne-Lyubeznik-Smith.

Question

Suppose that M is a finite R-module and that $\phi : M \to M$ is an additive map such that $\phi(r^{p^e}x) = r\phi(x)$. Let ϕ_n be the map obtained by composing ϕ with itself n-times. Does

$$\mathrm{Im}(\phi_1) \supseteq \mathrm{Im}(\phi_2) \supseteq \mathrm{Im}(\phi_3) \supseteq \dots \text{ stabilize?}$$

- If this is true, then one can modify the KLZ proof to work for any F-finite ring (not necessarily local).
- We can answer this question affirmatively for R of finite type over a perfect field.

The Blickle-Mustaţă-Smith argument

- In their proof, they use a characterization of the test ideal which uses the following construcdtion.
- Given an ideal I, they define $I^{[1/p^e]}$ to be the smallest ideal J of R such that $I \subseteq J^{[p^e]}$.
- However, this $[1/p^e]$ construction can be interepretted as a map $R^{1/p^e} \to R$. Thus this Δ gives a natural way to generalize their argument.
- One reduces to the regular case via "F-adjunction".

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal and essentially of finite type over a perfect field. Further suppose that $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of $(R, \Delta, \mathfrak{a}^t)$ is a discrete set of rational numbers.

The Blickle-Mustaţă-Smith argument

- In their proof, they use a characterization of the test ideal which uses the following construcdtion.
- Given an ideal I, they define $I^{[1/p^e]}$ to be the smallest ideal J of R such that $I \subseteq J^{[p^e]}$.
- However, this $[1/p^e]$ construction can be interepretted as a map $R^{1/p^e} \to R$. Thus this Δ gives a natural way to generalize their argument.
- One reduces to the regular case via "F-adjunction".

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal and essentially of finite type over a perfect field. Further suppose that $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of $(R, \Delta, \mathfrak{a}^t)$ is a discrete set of rational numbers.

The Blickle-Mustaţă-Smith argument

- In their proof, they use a characterization of the test ideal which uses the following construcdtion.
- Given an ideal I, they define $I^{[1/p^e]}$ to be the smallest ideal J of R such that $I \subseteq J^{[p^e]}$.
- However, this $[1/p^e]$ construction can be interepretted as a map $R^{1/p^e} \to R$. Thus this Δ gives a natural way to generalize their argument.
- One reduces to the regular case via "F-adjunction".

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal and essentially of finite type over a perfect field. Further suppose that $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of $(R, \Delta, \mathfrak{a}^t)$ is a discrete set of rational numbers.

The Blickle-Mustaţă-Smith argument

- In their proof, they use a characterization of the test ideal which uses the following construcdtion.
- Given an ideal I, they define $I^{[1/p^e]}$ to be the smallest ideal J of R such that $I \subseteq J^{[p^e]}$.
- However, this $[1/p^e]$ construction can be interepretted as a map $R^{1/p^e} \to R$. Thus this Δ gives a natural way to generalize their argument.
- One reduces to the regular case via "F-adjunction".

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal and essentially of finite type over a perfect field. Further suppose that $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of $(R, \Delta, \mathfrak{a}^t)$ is a discrete set of rational numbers.

The Blickle-Mustaţă-Smith argument

- In their proof, they use a characterization of the test ideal which uses the following construcdtion.
- Given an ideal I, they define $I^{[1/p^e]}$ to be the smallest ideal J of R such that $I \subseteq J^{[p^e]}$.
- However, this $[1/p^e]$ construction can be interepretted as a map $R^{1/p^e} \to R$. Thus this Δ gives a natural way to generalize their argument.
- One reduces to the regular case via "F-adjunction".

Theorem (S., Takagi, Zhang)

Suppose that (R, Δ) is a pair such that R is normal and essentially of finite type over a perfect field. Further suppose that $n(K_R + \Delta)$ is Cartier where p does not divide n. Then the set of F-jumping numbers of $(R, \Delta, \mathfrak{a}^t)$ is a discrete set of rational numbers.

Outline

de Fernex and Hacon's new multiplier ideals

- Recently, de Fernex and Hacon have introduced multiplier ideals for pairs $(X, \mathfrak{a}^t)$ when X is not $\mathbb{Q}$-Gorenstein (and there is no Δ).
- There still seem to be jumping numbers, and one can ask about discreteness and rationality there as well.
- However, it's completely open!
- Furthermore, the things one can prove about such multiplier ideals seem to coincide with what we know about test ideals.

de Fernex and Hacon's new multiplier ideals

- Recently, de Fernex and Hacon have introduced multiplier ideals for pairs $(X, \mathfrak{a}^t)$ when X is not $\mathbb{Q}$-Gorenstein (and there is no Δ).
- There still seem to be jumping numbers, and one can ask about discreteness and rationality there as well.
- However, it's completely open!
- Furthermore, the things one can prove about such multiplier ideals seem to coincide with what we know about test ideals.

de Fernex and Hacon's new multiplier ideals

- Recently, de Fernex and Hacon have introduced multiplier ideals for pairs $(X, \mathfrak{a}^t)$ when X is not $\mathbb{Q}$-Gorenstein (and there is no Δ).
- There still seem to be jumping numbers, and one can ask about discreteness and rationality there as well.
- However, it's completely open!
- Furthermore, the things one can prove about such multiplier ideals seem to coincide with what we know about test ideals.

de Fernex and Hacon's new multiplier ideals

- Recently, de Fernex and Hacon have introduced multiplier ideals for pairs $(X, \mathfrak{a}^t)$ when X is not $\mathbb{Q}$-Gorenstein (and there is no Δ).
- There still seem to be jumping numbers, and one can ask about discreteness and rationality there as well.
- However, it's completely open!
- Furthermore, the things one can prove about such multiplier ideals seem to coincide with what we know about test ideals.