A Geometric Characterization of (generalizations of) F-Ideals

Centers of F-purity

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Outline

1. Background and Definitions
   - Characteristic $p > 0$ singularities
   - Log Canonical Centers
   - Properties of log canonical centers
2. A Positive Characteristic Analogue of Log Canonical Centers
   - Definitions And First Properties
   - Uniformly F-Compatible Ideals
3. Applications
   - Deeper properties and results related to F-adjunction
   - New Results in Characteristic Zero

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Summary

- Link a notion from characteristic zero (called *log canonical centers / centers of log canonicity*), with (generalizations of) *annihilators of F-stable submodules of local cohomology*.
  - A submodule $N \subset H^d_{\mathfrak{m}}(R)$ is called *F-stable* if $F(N) \subseteq N$ under the Frobenius map $F : H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}(R)$.
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Universal Assumptions

- All rings are reduced.
- All rings of characteristic zero are essentially of finite type over $\mathbb{C}$ (where $\mathbb{C}$ is your favorite field of characteristic zero).
- All rings of positive characteristic are F-finite
  - (that is, if R is viewed as an R-module via the action of Frobenius, it is a finite R-module).
  - (in other words, $R^{\frac{1}{p}}$ is a finite R-module).

Some Definitions
F-pure rings and pairs

All rings are assumed to be F-finite of characteristic $p > 0$. Let $\mathfrak{a} \subset R$ be an ideal with $\mathfrak{a} \cap R^{\circ} \neq 0$ and suppose that $t > 0$ is a positive number.

- A ring is called *F-pure* if the Frobenius map $R \to R^{\frac{1}{p^e}}$ splits.
- A pair $(R, \mathfrak{a}^t)$ is called *sharply F-pure* if there exists an integer $e > 0$ and an $a \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$ such that the map

$$R \longrightarrow R^{\frac{1}{p^e}} \xrightarrow{\times a^{\frac{1}{p^e}}} R^{\frac{1}{p^e}}$$

splits.

There exist definitions for pairs (R, Δ) also.

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Same assumptions as before.

- A ring R is called *strongly F-regular* if for every $c \in R^{\circ}$, there exists an $e > 0$ such that the map

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- Given an ideal $I \subset R$, the *$\mathfrak{a}^t$-tight closure of I*, (denoted $I^{*\mathfrak{a}^t}$) is defined as

$$\{x \in R \mid \exists c \in R^{\circ}, \text{such that for all } e \gg 0, c\mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e} \in I^{[p^e]}\}.$$

- Given a module $N \subseteq M$, the *$\mathfrak{a}^t$-tight closure of N in M*, (denoted $N_M^{*\mathfrak{a}^t}$) is defined as

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Here z^{p^e} is defined to be the image of z via the map $M \to R^{\frac{1}{p^e}} \otimes_R M$ and $N_M^{[p^e]}$ is defined to be the image of $R^{\frac{1}{p^e}} \otimes_R N$ inside $R^{\frac{1}{p^e}} \otimes_R M$.

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Even More Definitions

Test Ideals

We keep on with the same assumptions.

- The *test ideal* $\tau(\mathfrak{a}^t)$ is defined to be the set of elements $c \in R$ such that for every $x \in I^{*\mathfrak{a}^t}$ we have that
$$c\mathfrak{a}^{\lceil t(p^e-1)\rceil} x^{p^e} \in I^{[p^e]}$$
for all $e \geq 0$.
- The *big/non-finitistic test ideal* $\widetilde{\tau}(\mathfrak{a}^t) = \tau_b(\mathfrak{a}^t)$ is defined to be the set of elements $c \in R$ such that for every $z \in N_M^{*\mathfrak{a}^t}$ we have that
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The Final Definition
in positive characteristic

- A ring R is said to be *F-injective* if for every maximal ideal $\mathfrak{m} \in m - \operatorname{Spec} R$, the induced Frobenius map on local cohomology
$$H^i_{\mathfrak{m}}(R_{\mathfrak{m}}) \to H^i_{\mathfrak{m}}(R_{\mathfrak{m}})$$
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A Review of the Dictionary

F-singularities vs singularities in birational geometry

- Begin with a pair (R, Δ) where R is a normal domain of finite type over $\mathbb{C}$ and Δ is an effective $\mathbb{Q}$-divisor on $X = \operatorname{Spec} R$
  - (Δ is a formal sum prime divisors on X with nonnegative rational coefficients.)
- Assume that $K_X + \Delta$ is $\mathbb{Q}$-Cartier, if $\Delta = 0$ this means R is $\mathbb{Q}$-Gorenstein
  - (For some integer $n > 0$, $\mathcal{O}_X(n(K_R + \Delta))$ is a locally free)
- Reduce generically to characteristic p.

Positive Characteristic		Characteristic Zero
Test Ideals, $\tau(\Delta)$	$\Longleftrightarrow$	Multiplier Ideals, $\mathcal{J}(\Delta)$
F-Pure Singularities	$\Longrightarrow$	Log Canonical Singularities
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An Informal Definition of Log Canonical Centers

Suppose that (R, Δ) is log canonical. (If you want to assume that there is no Δ, that's ok)

- Roughly speaking, we say that $Q \in \operatorname{Spec} R$ (not necessarily a maximal ideal) is a *log canonical center* if
- the pair (R_Q, Δ_Q) is only "barely" log canonical
  - (R_Q, Δ_Q) is just the pair (R, Δ) localized at Q.

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The Definition of Centers of Log Canonicity

aka Log Canonical Centers

Choose $Q \in \operatorname{Spec} R$. We say that Q is a

center of log canonicity / log canonical center /non-log terminal center

for the pair (R, Δ), *IF*

- For every element $f \in Q$,
- and every $\epsilon > 0$,
- the pair $(R, \Delta + \epsilon \operatorname{div}(f))$ is *NOT* log canonical at Q (at the local ring/stalk),

  this is the same as

  - the triple $(R_Q, \Delta_Q, f^\epsilon)$ is *NOT* log canonical
  - if $\Delta = 0$, this is just the same as saying that (R_Q, f^ϵ) is *NOT* log canonical

If you are comfortable with "discrepancies", the log canonical centers are exactly the points $Q \in \operatorname{Spec} R$ which are dominated by divisors with discrepancy ≤ -1.

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Basic Observations About Log Canonical Centers

Suppose that (R, Δ) is log canonical.

- There are only finitely many log canonical centers.
  - This follows since the log canonical centers can be identified on a single resolution (using the description involving discrepancies)
- The intersection of all centers of log canonicity is the multiplier ideal.
  - This follows since in a log canonical pair, the multiplier ideal is a radical ideal and defines the non-log terminal locus.

Recall:

Positive Characteristic		Characteristic Zero
Test Ideals	$\Longleftrightarrow$	Multiplier Ideals
F-Pure Singularities	$\Longrightarrow$	Log Canonical Singularities

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Deeper Results and Relations To Characteristic $p > 0$

Characteristic Zero (R, Δ) is log canonical	**Characteristic $p > 0$** R is F-pure
Finite # of log canonical centers.	If R is local, finite # of annihilators of F-stable submodules of $H^d_{\mathfrak{m}}(R)$ [Enescu-Hochster, Sharp]
If R is log terminal then $R/\mathcal{J}(\Delta)$ is Du Bois [-]	$R/\tau(R)$ is F-pure [Vassilev], also see [Fedder-Watanabe]
If R is log terminal then $R/$ (a largest log canonical center) is log terminal [Kawamata]	If R is local then $R/$ (Splitting Prime) is strongly F-regular, [Aberbach-Enescu]
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Deeper Results and Relations To Characteristic $p > 0$

Characteristic Zero (R, Δ) is log canonical	**Characteristic** $p > 0$ R is F-pure
Finite # of log canonical centers.	If R is local, finite # of annihilators of F-stable submodules of $H^d_{\mathfrak{m}}(R)$ [Enescu-Hochster, Sharp]
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Outline

A First Definition Of Centers of F-Purity

Definition. Suppose that R is a reduced F-finite ring. We say that $Q \in \operatorname{Spec} R$ is a *center of F-purity* if

- For every $f \in QR_Q$ and for every $e \geq 0$,
- the map

$$\begin{array}{ccc} R_Q & \xrightarrow{\alpha} & R_Q^{\frac{1}{p^e}} \\ 1 & \longmapsto & f^{\frac{1}{p^e}} = \alpha(1) \end{array}$$

does *NOT* split.

Note that any minimal prime of the non-strongly F-regular locus is a center of F-purity.

If you wish to work with triples $(R, \Delta, \mathfrak{a}^t)$, simply replace $R_Q^{\frac{1}{p^e}}$ with $(R(\lceil (p^e - 1)\Delta \rceil))_Q^{\frac{1}{p^e}}$, you also want $1 \longmapsto (af)^{\frac{1}{p^e}}$ not to split for every $a \in \mathfrak{a}^{\lceil t(p^e-1) \rceil}$. This provides the notion of *centers of sharp F-purity*.

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An Example

Suppose that k is a perfect field of characteristic $p > 0$. Consider the ring

$$R = k[a, b, c]/(a^3 + abc - b^2) = k[xy, x^2y, x - y] \subset k[x, y]$$

It is easy to verify that this ring is F-pure [Fedder].

Its centers of F-purity are exactly the ideals (0), (a, b) and (a, b, c).

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A Characterization of Centers of F-Purity

Proposition. *Suppose that R is a reduced F-finite ring and that $Q \in \operatorname{Spec} R$. Then the following are equivalent:*

(1) *Q is a center of F-purity.*

(2) *For every $e > 0$ and every map $\phi \in \operatorname{Hom}_{R_Q}({R_Q}^{1/p^e}, R_Q)$, $\phi((QR_Q)^{\frac{1}{p^e}}) \subseteq QR_Q$.*

(3) *For every $e > 0$ and every map $\phi \in \operatorname{Hom}_R(R^{1/p^e}, R)$, $\phi(Q^{\frac{1}{p^e}}) \subseteq Q$.*

Proof.

(1) $\leftrightarrow$ (2): note that "not splitting" is basically not sending elements of Q to units. (2) $\leftrightarrow$ (3) is straightforward. □

- This generalizes to triples $(R, \Delta, \mathfrak{a}^t)$.

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A Better Definition to Work With?

If at first you don't succeed re"define" success...

Definition. Suppose R is a reduced F-finite ring. We say that an ideal $I \subset R$ is *uniformly F-compatible* if

- for every $e > 0$ and
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I'd also like to state this definition for triples $(R, \Delta, \mathfrak{a}^t)$ (where $\mathfrak{a} \subseteq R$ is a non-zero ideal and $t > 0$ is a rational number).

Definition. Suppose R is a normal F-finite ring. We say that an ideal $I \subset R$ is *uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible* if for every $e > 0$ and for every $\phi \in \mathrm{Hom}_R(R(\lceil (p^e - 1)\Delta \rceil)^{\frac{1}{p^e}}, R)$, we have

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First Observations About Uniformly F-Compatible Ideals

Consider the following properties of uniformly F-compatible ideals (analogues hold for various sorts of pairs and triples).

(1) Any intersection of uniformly F-compatible ideals is uniformly F-compatible.

(2) Any sum of uniformly F-compatible ideals is uniformly F-compatible.

(3) Any radical of a uniformly F-compatible ideal is uniformly F-compatible.

(4) Any associated prime of a radical uniformly F-compatible ideal is uniformly F-compatible.

(5) In an F-pure ring if I is uniformly F-compatible, then R/I is F-pure. In particular, I is radical.

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(2) Any sum of uniformly F-compatible ideals is uniformly F-compatible.
(3) Any radical of a uniformly F-compatible ideal is uniformly F-compatible.
(4) Any associated prime of a radical uniformly F-compatible ideal is uniformly F-compatible.
(5) In an F-pure ring if I is uniformly F-compatible, then R/I is F-pure. In particular, I is radical.
(6) A prime Q is uniformly F-compatible if and only if Q is a center of F-purity.

Finitely Many Uniformly F-Compatible Ideals

The various properties on the previous page, together with the techniques of [Enescu-Hochster] imply the following

Corollary. *If $(R, \mathfrak{m})$ is an F-finite local ring and $(R, \Delta, \mathfrak{a}^t)$ is a sharply F-pure triple, then there are at most finitely many uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible ideals.*

This can also be obtained in the non-pair case by the techniques of [Sharp].

It does however suggest the following question.

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Alternate Characterizations of Uniformly F-Compatible Ideals

Suppose that $(S, \mathfrak{m})$ is an F-finite regular local ring and that $R = S/I$ is a quotient. Suppose that $J' \subset S$ is an ideal containing I and set $J = J'/J \subset R$. TFAE:

(a) J is uniformly F-compatible.

(b) For every $e > 0$ and every $f \in J$, the composition

$$\mathrm{Ann}_{E_R}(J) = E_{R/J} \longrightarrow E_R \longrightarrow R^{\frac{1}{p^e}} \otimes_R E_R \xrightarrow{\times f^{\frac{1}{p^e}}} R^{\frac{1}{p^e}} \otimes_R E_R$$

is zero.

(c) For every $e > 0$ we have $(I^{[p^e]} : I) \subseteq (J'^{[p^e]} : J')$.

(d) $\mathrm{Ann}_{E_R}(J)$ is an $\mathcal{F}(E_R)$-submodules of E_R. [Lyubeznik-Smith]

Condition (b) generalizes to triples $(R, \Delta, \mathfrak{a}^t)$, condition (c) generalizes to pairs $(R, \mathfrak{a}^t)$.

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Outline

1. Background and Definitions
   - Characteristic $p > 0$ singularities
   - Log Canonical Centers
   - Properties of log canonical centers
2. A Positive Characteristic Analogue of Log Canonical Centers
   - Definitions And First Properties
   - Uniformly F-Compatible Ideals
3. Applications
   - Deeper properties and results related to F-adjunction
   - New Results in Characteristic Zero

Many Common Ideals are Uniformly F-Compatible

Suppose that R is an F-finite reduced ring.

- The test ideal $\tau(R)$ is uniformly F-compatible.
- If R is F-pure, then the conductor is uniformly F-compatible.
- If I is an annihilator of any F-stable submodule of $H^d_{\mathfrak{m}}(R)$, then I is uniformly F-compatible.
- If R is F-pure and local, then the splitting prime $\mathcal{P}$, see [Aberbach-Enescu], is uniformly F-compatible
  - In fact, in this case, the splitting prime is the unique largest uniformly F-compatible proper ideal.

The previous results also hold for pairs/triples (where appropriate)

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- The non-finitistic/big test ideal $\widetilde{\tau}(R) = \tau_b(R)$ is uniformly F-compatible.
  - In fact, it is the smallest uniformly F-compatible ideal that contains an element of R°.
- If (R, Δ) is a pair and $\pi : Y \to \operatorname{Spec} R$ is a proper birational map with Y normal, then for any effective divisor G on Y such that $\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_R + \Delta) + G \rceil) \subseteq \mathcal{O}_{\operatorname{Spec} R}$, we have that (the global sections of) $\pi_* \mathcal{O}_Y(\lceil K_Y - \pi^*(K_R + \Delta) + G \rceil)$ is uniformly (Δ, F)-compatible.
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Additional Relations to Test Ideals

The relation to the test ideal (mentioned on the previous page) should not be surprising.

- In [Lyubeznik-Smith], the authors showed that if $R = S/I$ is a domain where S is a regular local ring, then the big test ideal $\widetilde{\tau}_R$ corresponds to the smallest ideal $J \supsetneq I$ which satisfies $(I^{[p^e]} : I) \subseteq (J^{[p^e]} : J)$ for all $e > 0$.
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Results Related to F-Adjunction

Some of these have been mentioned before, but I'd like to repeat them. Suppose that $(R, \Delta, \mathfrak{a}^t)$ is a triple.

- If I is uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible, and if $J \supset I$ corresponds to to a uniformly F-compatible ideal of R/I, then J is also uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible.
- If I is a proper uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible ideal that is maximal with respect to containment (among uniformly compatible ideals), then R/I is strongly F-regular.
  - If $(R, \Delta, \mathfrak{a}^t)$ is local but not sharply F-pure, then the maximal such ideal with respect to containment is $\mathfrak{m}$.
- If I is uniformly $(\Delta, \mathfrak{a}^t, F)$-compatible and $(R, \Delta, \mathfrak{a}^t)$ is sharply F-pure, then R/I is F-pure.
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R/I is of Dense F-Pure Type

In fact, the result does lift. Suppose that R is a ring of finite type over $\mathbb{C}$.

Theorem. *Suppose that (R, Δ) has dense F-pure type (which is conjecturally equivalent to being log canonical) and I is an intersection of centers of log canonicity (a scheme-theoretic union). Then R/I also has dense F-pure type (in particular it has Du Bois singularities).*

Proof.

Reduce (R, Δ) together with I and a log resolution, to a family of characteristic $p > 0$ models. The ideals $\{I_p\}$ in the family corresponding to I are uniformly (Δ_p, F)-compatible (this can be seen in several ways). Thus R_p/I_p is F-pure for an infinite set of p. □

Happy Birthday Mel

Happy Birthday Mel![1]

[1] two days belated Happy Birthday Mel!