Chapter 3
SEMINORMALITY

In this section we still restrict ourselves to noetherian rings, even though much of this work can be generalized to broader cases.

Definition 3.5. [GT80] A ring is said to be a *Mori Ring* if its integral closure is a finite extension.

We will restrict to only Mori rings throughout this section. Generalizations of seminormality without this hypothesis exist, but we will not explore them.

We will follow the definition of seminormality given in [GT80] and [Tra70] which have a more geometric flavor. Seminormality can be defined in more general situations as well, see [Swa80]. Also see [LV81a] and [RRS96].

Definition 3.6. [GT80], [Swa80] An integral extension of rings $i : A \subset B$ is said to be subintegral if it induces a bijection on the prime spectra, and if for every prime $P \in \operatorname{Spec} B$, the induced maps on the residue fields, $k(i^{-1}(P)) \to k(P)$, is an isomorphism.

Remark 3.7. This sort of extension of rings was called a quasi-isomorphism in [GT80]

Definition 3.8. An integral extension of rings $i : A \subset B$ is said to be weakly subintegral if it induces a bijection on the prime spectrums, and if for every prime $P \in \operatorname{Spec} B$, the induced maps on residue fields $k(i^{-1}(P)) \to k(P)$ is a purely inseparable extension of fields.

When working over rings that contain a copy of $\mathbb{Q}$ the notions of subintegral and weakly subintegral clearly coincide. In this paper, when dealing with weakly subintegral extensions, we will most often be considering rings that contain a copy of a (finite) field and not the mixed characteristic case.

From subintegrality and weak subintegrality come the notions of seminormality and weak normality, which are essentially the notions of subintegral closure (respectively weak subintegral closure) of a ring extension.

Definition 3.9. [GT80, 1.2], [Swa80, 2.2] Let $A \subset B$ be an extension of rings. Define ${}^{+}_{B}A$ to be the (unique) largest subextension of A in B such that $A \subset {}^{+}_{B}A$ is subintegral. A ring A is said to be seminormal in B if $A = {}^{+}_{B}A$

Definition 3.10. [Yan85], [RRS96, 1.1] Let $A \subset B$ be an extension of rings. Define ${}^{*}_{B}A$ to be the (unique) largest subextension of A in B such that $A \subset {}^{*}_{B}A$ is weakly subintegral. A ring A is said to be weakly normal in B if $A = {}^{*}_{B}A$.

One should note that if $b \in B$ and $A \subset B$ is subintegral, $A \subset A[b]$ is also subintegral, see 3.15 below. With this in mind, we say that $b \in B$ is subintegral (respectively weakly subintegral) over A if $A[b]$ is subintegral (respectively weakly subintegral) over A. It is also true that ${}^{+}_{B}A$ (respectively ${}^{*}_{B}A$) is the set of all elements of B that are subintegral (respectively weakly subintegral) over A.

Definition 3.11. A reduced ring A is said to be seminormal (respectively weakly normal) if it is seminormal (respectively weakly normal) in its integral closure $\overline{A}$ (of its total field of fractions). Its seminormalization (respectively weak normalization) is ${}^{+}_{\overline{A}}A$ and will be denoted by ${}^{+}A$ (respectively ${}^{*}_{\overline{A}}A$ and will be denoted as ${}^{*}A$). If $X = \operatorname{Spec} A$ is a scheme, X^{sn} will be used to denote the scheme $\operatorname{Spec} {}^{+}_{\overline{A}}A$ and X^{wn} will be used to denote the scheme $\operatorname{Spec} {}^{*}_{\overline{A}}A$

Geometrically speaking, seminormality means that any non-normality (read gluing) of a scheme will be as transverse as possible. Let us include several examples for the convenience of the reader unfamiliar with these notions.

Example 3.12 (Curves)**.** It is easy to see that $k[x^2, x^3] \subset k[x]$ is a subintegral, extension and thus a cuspidal curve is not seminormal (it induces a bijection on points). Therefore the seminormalization of a cuspidal curve is just the normalization. On the other hand, let us consider a nodal curve. It is clear that $k[x(x-1), x^2(x-1)] \subset k[x]$ is not subintegral and it is easy to see that it has no proper subextension that is. The general principle is this: whenever one glues points of an integrally closed domain in such a way as to obtain the most transverse intersections possible, one obtains a seminormal ring. From this point of view we do see that a tacnode [Har77, Chapter I, Section 5] is not seminormal, and its seminormalization is a node. It is also interesting to note that three lines intersecting at the origin in $\mathbb{A}^2_k$ is not seminormal but the three coordinate axes intersecting in $\mathbb{A}^3_k$ is seminormal. More generally, if one works over an algebraically closed field of characteristic zero, seminormality in dimension 1 is equivalent to the so-called multicross singularity [LV81b]. Recall that $x \in X$ is called a multicross singularity if (X, x) is analytically isomorphic to (X', x') where X' is the union of linear subspaces of affine space meeting transversally along a common linear subspace.

Example 3.13 (A Pinch Point)**.** The pinch point is given by the ring $k[x, xy, y^2] \subset k[x, y]$. It is obtained by taking one of the coordinate axes $\mathrm{Spec}(k[x, y]/(x) = k[y])$ of $\mathbb{A}^2_k = \mathrm{Spec}\, k[x, y]$ and gluing its points together in a two to one fashion as in the inclusion $k[y^2] \subset k[y]$. The "pinched point" is at the ramified point of the gluing. The pinch point is always seminormal and is weakly normal except when the characteristic of k is two. In characteristic two, one obtains a bijection of points and, on the closed points, an isomorphism of residue fields. However, on the general point of the glued axis one observes the extension $k(y^2) \subset k(y)$ which is purely inseparable and not an isomorphism.

Example 3.14 (Replacing Residue Fields)**.** Consider the inclusion $\mathbb{R}[x, ix] \subset \mathbb{C}[x]$. This inclusion of rings (of characteristic zero) induces a bijection on points of the Spec's, and induces an isomorphism of residue fields at every point except the origin (where it induces $\mathbb{R} \subset \mathbb{C}$). Thus it is not a subintegral extension (and it is easy to see that $\mathbb{R}[x, ix]$ is seminormal).

Likewise, let k be a perfect field of characteristic p. Consider the extension

$$S = k(x^p)[y, yx, yx^2, \ldots, yx^{p-1}] \subset k(x)[y].$$

Again it is easy to see that this injection induces a bijection of points and an isomorphism of residue fields except at the origin (where we obtain $k(x^p) \subset k(x)$. It is easy to see that S is seminormal but not weakly normal.

The following facts about seminormality (and weak normality) are very useful.

Lemma 3.15. *[Swa80, 2.3] Let $A \subset B \subset C$ be an extension of rings. Then $A \subset C$ is* *subintegral (respectively weakly subintegral) if and only if $A \subset B$ and $B \subset C$ are subintegral also.*

Proposition 3.16. *[LV81a, 1.4] Let $A \subset B$ be an integral extension of rings; the following are then equivalent:*

(i) A is seminormal in B

(ii) A contains each element of $b \in B$ such that $b^n, b^{n+1} \subset A$ for some positive integer $n > 1$.

(iii) For a fixed pair of relatively prime integers $e > f > 1$, A contains each element $b \in B$ such that $b^e, b^f \in A$. (also see [Ham75] and [Swa80] for the case where $e = 2$, $f = 3$).

The following is another useful theorem. We say that an extension $A \subset A[b] \subset B$, $b \in B$, is an elementary subintegral extension of A in B if $b^2, b^3 \in A$.

Proposition 3.17. *[Swa80, 2.8] Let $A \subset B$ be any extension. Then ${}^+_BA$ is the filtered union of all subrings of B which can be obtained by a finite number of elementary subintegral extensions.*

In particular, if $A \subset B$ is finite, one needs only take a finite number of elementary subintegral extensions of A in B.

Proposition 3.18. *[RRS96, 4.3, 6.8] Let $A \subset B$ be an integral extension of rings where A contains $\mathbb{F}_p$ for some prime p; the following are then equivalent:*

(i) A is weakly normal in B.

(ii) If $b \in B$ and $b^p \in A$ then $b \in A$.

We also have the following result analogous to 3.17. If $A \subset B$ is an integral extension of rings, A contains a copy of $\mathbb{F}_p$ and $b \in B$. We say that $A[b]$ is elementary weakly subintegral over A if $b^p \in A$. Note that if a ring contains a field of characteristic p, then every elementary weakly normal extension is an elementary subintegral extension.

Proposition 3.19. *[Yan85, Theorem 2], [RRS96] Let $A \subset B$ be any extension of rings containing $\mathbb{F}_p$. Then ${}^*_BA$ is the filtered union of all subrings of B which can be obtained by a finite number of elementary weakly subintegral extensions.*

Seminormality has many remarkable properties. Seminormality can be made sense of intrinsically without appeal to an integral extension. That is, R is seminormal if and only if it is reduced, and whenever $b, c \in R$ satisfy $b^3 = c^2$, there exists an $a \in$ with $a^2 = b$ and $a^3 = c$. [Swa80]. One should note that weak normality cannot be nearly so intrinsic since $\mathbb{Z}_p[x]$ is a weakly subintegral extension of $\mathbb{Z}_p[x^p]$ even though both rings are integrally closed in their own fraction fields. Seminormalization also commutes with localization [LV81a, 1.6], [GT80, 2.9] and is functorial in the following sense.

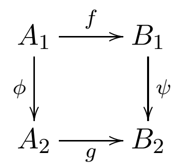
Theorem 3.20. *[Swa80, 4.1] Let A be any reduced commutative ring. Then there is a subintegral extension $A \subset B$ with $B = {}^+A$ seminormal. Any such extension is universal for maps of A to seminormal rings: If C is seminormal and $\phi : A \to C$, then ϕ has a unique extension $\psi : B \to C$. Furthermore, ψ is injective if ϕ is.*

This universal property implies that B is unique up to canonical isomorphism and is equal to its seminormalization (as defined above) up to isomorphism. In particular, if $A \to D$ is a map of reduced rings, then there is a unique induced map ${}^+A \to {}^+D$. This is not true for normalization unless the rings are domains and the map is an inclusion. Consider the following example:

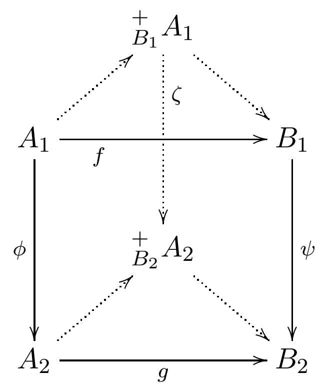
Example 3.21. Consider the map $\phi : R = k[x, y]/(xy) \to k \oplus k[x] \oplus k[y] = S$ which sends x to $(0, x, 0)$ and y to $(0, 0, y)$ and 1 to $(1, 1, 1)$. This is basically the normalization of $k[x, y]/(xy)$ with an additional disconnected point mapping to where the two lines cross. Note that ϕ is injective, S is already integrally closed and the integral closure of R is $k[x, y]/(y) \oplus k[x, y]/(x) = \overline{R}$. Geometrically it is obvious that we cannot extend the original map to a map between these rings in a canonical way, since there are two possible places to map the disconnected point.

Seminormality is also functorial in the following sense. We provide an (alternate) proof merely for completeness.

Proposition 3.22. *[Amb98, 4.2] Suppose $f : A_1 \to B_1$ and $g : A_2 \to B_2$ are extensions of reduced rings. If there are finite maps $\phi : A_1 \to A_2$ and $\psi : B_1 \to B_2$ as below*



making the diagram commute, then there is a unique map $\zeta : {}^{+}_{B_1}A_1 \to {}^{+}_{B_2}A_2$ making the following diagram commute:

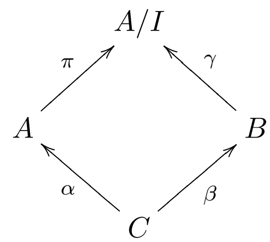


Alternately, the $^{+}$'s may be replaced by *'s and the same statement holds.

Proof. Begin by restricting ψ to ${}^{+}_{B_1}A_1$. This provides a map $\zeta' : {}^{+}_{B_1}A_1 \to B_2$. We need to show that the image of ζ' is made up of elements subintegral over B_2. Choose an element $a_1 \in {}^{+}_{B_1}A_1$ such that $A[a_1]$ is an elementary subintegral extension of A. That means $a_1^2, a_1^3 \in A$. In particular, $\psi(a_1)^2$ and $\psi(a_1)^3$ are contained in A_2, and so $\psi(a_1)$ is contained in ${}^{+}_{B_2}A_2$. We now choose $a_2 \in {}^{+}_{B_1}A_1$ now elementary subintegral over $A_1[a_1]$ and repeat the process proving that $\psi(a_2)$ is elementary subintegral over $A_2[\psi(a_1)]$ and thus subintegral over A_2 as well [Swa80, 2.3]. Since the maps are ϕ and ψ finite, this process eventually stops. □

We now move in a somewhat more geometric direction. The following is actually just [GT80, 4.3] with the weak normality hypothesis as well. We provide an alternate proof for the convenience of the reader. Also compare with [Swa80, 3.3].

Lemma 3.23. *[GT80, 4.3] Suppose A is a ring, $I \subset A$ a reduced ideal, B another ring and $\gamma : B \to A/I$ a finite extension of rings. Then if γ is a seminormal extension (or a weakly* *normal extension, in which case we assume all rings contain a finite field of characteristic p), so is α.*



where $C = \{(a, b) \in A \oplus B | \pi(a) = \gamma(b)\}$ is the pullback in the category of rings and α and β are the projections.

Proof. It is easy to see that since γ is injective, so is α. Now suppose that there exist $a \in A$ such that a^2 and a^3 are in the image of α (or, if we are supposing weakly subintegral, we assume there exists $a \in A$ such that a^p is in the image of α). Thus there exists b_2 and b_3 in B so that $(a^2, b_2), (a^3, b_3) \in C$ (or there exists $b_p \in B$ so that $(a^p, b_p) \in C$). There are two cases; the first is that a^2 (or a^3 or a^p) is in I. But then the b_i's are zero since γ is injective and $a \in I$ as well, since I is reduced. In that case $(a, 0) \in C$ and $(a, 0)^i = (a^i, b_i)$. On the other hand, let us assume that none of the powers of a are in I. This means that $\gamma(b_2) = \pi(a)^2$ and $\gamma(b_3) = \pi(a)^3$. But then there exists a unique $b \in B$ so that $b^i = b_i$ and $\gamma(b) = \pi(a)$ since γ is subintegral (respectively weakly subintegral). Then $(a, b) \in C$ as desired. □

The following gluing lemma(s) are proven in [Sch05] (for the category of algebraic spaces see [Art70, 6.1]). Here $X \cup_Z Y$ is the fibered coproduct in the category of ringed spaces (that is, take the pushout in the category of topological spaces and the pullback in the category of rings).

Theorem 3.24. *[Sch05, 3.4] Suppose A and B are rings. Further suppose I is an ideal of A and there exists a map γ from B to A/I. We will denote the quotient map from A to A/I by π. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} A/I$, so that Z is a closed subscheme of X. Then $X \cup_Z Y$ is an affine scheme with Y a closed subscheme, $(X \cup_Z Y) - Y \cong X - Z$, and the maps $\alpha : X \to X \cup_Z Y$ and $\beta : Y \to X \cup_Z Y$ are morphisms of schemes.*

Roughly speaking we glue Y into X along Z.

Theorem 3.25. *[Sch05, 3.5] The scheme constructed in 3.24 is a pushout (fibered coproduct) in the category of schemes.*

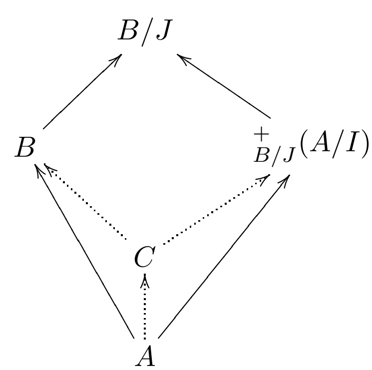
The upshot of these theorems is that the fibered coproduct in the category of schemes is just the Spec of pullback of rings. Note that it is possible that the pullback of rings is not noetherian.

We also need a version of the following well known fact:

Lemma 3.26. *Let $f : A \to B$ be a map of (noetherian) rings with A reduced and such that the corresponding map $\phi : \operatorname{Spec} B \to \operatorname{Spec} A$ has a dense image in the Zariski topology; then f is injective.*

Proof. Suppose that $0 \neq a \in A$ and consider $V(a) \subset \operatorname{Spec} A$. Since A is reduced this means that $V(a)$ is strictly smaller than $\operatorname{Spec} A$. In particular, $\operatorname{Spec} A - V(a) = \operatorname{Spec}(A[a^{-1}])$ is a nonempty open set of $\operatorname{Spec} A$ and so there exists a prime ideal $P \in \operatorname{Spec} B$ such that $\phi(P) \in \operatorname{Spec} A - V(a) \subset \operatorname{Spec} A$. This implies that $\phi(P) = f^{-1}(P)$ does not contain a and so $f(a) \neq 0$. $\square$

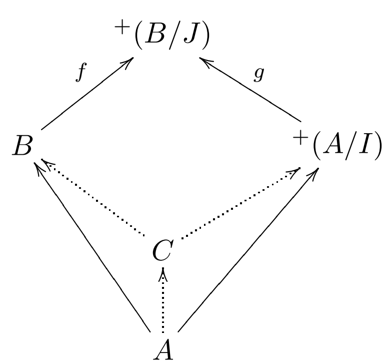
Proposition 3.27. *Let A be a ring, $X = \operatorname{Spec} A$ and let $A \subset B$ be an extension of reduced rings. Let $Y = \operatorname{Spec} B$ and further suppose that $\phi : Y \to X$ is a finite map (in particular it surjects on the* Spec*'s). Let $I \subset A$ be a reduced ideal containing the locus over which ϕ is not an isomorphism. Let $J \subset B$ be the radical of IB (note $A/I \subset B/J$ by 3.26). Then the pullback C of the diagram*



is the seminormalization of A in B. Alternately, if one replaces $+$ by $*$ in the above setup (and assume all rings involved contain a field of characteristic p) then C becomes the weak normalization of A in B.

Proof. All we need to establish here is that the map $A \to C$ is subintegral by 3.23. It is clear that $A \subset C \subset B$ since $A \subset B$ and $A/I \subset B/J$. Since $A \subset B$ is a finite extension, so is $A \subset C$. We will do this using the original definition 3.6. The gluing theorem 3.24 guarantees a bijection of points of the prime spectra. There are two sorts of primes of C. First consider those coming from $\operatorname{Spec} B - V(J) = \operatorname{Spec} A - V(I)$ over which we clearly have a trivial residue field extension since $\operatorname{Spec} B - V(J) = \operatorname{Spec} C - V(J)$. On the other hand there are those coming from $\operatorname{Spec} {}^{+}_{B/J}(A/I)$ which is a closed subscheme of $\operatorname{Spec} C$. But those primes have trivial residue field extension over A since $\operatorname{Spec} {}^{+}_{B/J}(A/I)$ is subintegral over A/I and points of $\operatorname{Spec} A/I$ have the same residue fields as the corresponding points of $\operatorname{Spec} A$. ∎

Corollary 3.28. *Let A be a ring $X = \operatorname{Spec} A$ and let $A \subset B$ be a birational extension of reduced rings. Let $Y = \operatorname{Spec} B$ and further suppose that $\phi : Y \to X$ is a finite map. Let $I \subset A$ be a reduced ideal containing the locus over which ϕ is not an isomorphism (and assume $I \neq A$). Let $J \subset B$ be the radical of IB (note $A/I \subset B/J$ by 3.26). Then the pullback C of the diagram*



is the seminormalization of A in B.

Proof. Recall again that $C = \{(b, a') \in B \oplus {}^{+}(A/I) | f(b) = g(a')\}$. Then note that the image of B in ${}^{+}(B/J)$ is just B/J. We want to see that the a' of the pairs $(b, a') \in C$ that appear above are precisely the elements of ${}^{+}_{B/J}(A/I)$. First note that ${}^{+}_{B/J}(A/I) \subset {}^{+}_{{}^{+}(B/J)}(A/I) = {}^{+}(A/I)$ so that we only need the other inclusion. Choose $a' \in {}^{+}(A/I)$ $(b, a') \in C$. Let $\overline{b}$ denote the image of b in $B/J \subset {}^{+}(B/J)$. Then we see that $a' \in {}^{+}_{B/J}(A/I)$ as desired since

$A/I[\overline{b}] = A/I[a']$ is subintegral over A. Thus we see that pullback here is exactly the same as the pullback in 3.27, which completes the proof. □

This gives us a recursive but constructive algorithm to build the seminormalization a ring A. To construct the seminormalization of a ring A, first construct the normalization $\overline{A}$. Let I be the reduced ideal defining the non-normal locus of A, and $J = \sqrt{I\overline{A}}$. Now for the recursive step, construct the seminormalizations of A/I and A/J (note that these have strictly smaller dimension than $\dim A$). Then pull back the diagram of rings as above.

This corollary is also particularly useful in constructing seminormal varieties geometrically. The equations from the examples 3.12, 3.13, and 3.14 can all easily be obtained by using these sorts of gluing methods.

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