WHY ROW REPLACEMENT DOESN'T CHANGE THE SOLUTION SET

A couple of you have asked about a rigorous explanation of why this is true. Here's how I would explain it.

Suppose we are given a system of linear equations in n variables and m equations,

$$a_{11}x_1 + \ldots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + \ldots + a_{2n}x_n = b_2$$
$$\ldots$$
$$a_{m1}x_1 + \ldots + a_{mn}x_n = b_m$$

For shorthand, lets call this system of equations S. We want to show that it has the same solution as the system obtained by adding c times the ith row of S to the jth row of S. (i and j are arbitrary integers between 1 and m). We call this new, row-replaced system S'.

There are two things to show. Every solution of S is a solution of S' *AND* every solution of S' is a solution of S.

To start with, let $t = (t_1, \ldots, t_n)$ be a solution of S. We want to show it is also a solution of S'. Note that

$$a_{k1}t_1 + \ldots + a_{kn}t_n = b_k \tag{1}$$

holds for every value of k, and so it is easy to see that t is a solution for every equation of S' (except possibly the jth one, ie. the modified one). The jth equation of S' is

$$c(a_{i1}x_1 + \ldots + a_{in}x_n) + a_{j1}x_1 + \ldots + a_{jn}x_n = cb_i + b_j.$$

However, if we substitute the t's into the x's, using (1), we get that

$$cb_i + b_j = cb_i + b_j,$$

which is certainly true. Therefore, every solution of S is also a solution of S'.

To show the converse (that is, to show that every solution of S' is a solution of S) we note the following fact. One can do a single row replacement operation to S' to re-obtain S (add $-c$ times row i to row j). Thus we can perform the same argument as above (with the roles of S and S' reversed) and see that every solution of S' is also a solution of S.