MATH 217-4, QUIZ #7

1. Let V be a vector space and suppose that $S = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is a spanning set for V. We are also given a vector space W, and $\mathbf{w_1}, \mathbf{w_2}, \ldots \mathbf{w_n}$. Prove that there is at *MOST* one linear transformation $T : V \to W$ such that $T(\mathbf{v_1}) = \mathbf{w_1}, T(\mathbf{v_2}) = \mathbf{w_2}, \ldots,$ and $T(\mathbf{v_n}) = \mathbf{w_n}$. (7 points)

Suppose T and T' are linear transformations such that $T(\mathbf{v_i}) = T'(\mathbf{v_i})$ for all i. Choose $\mathbf{x}$ in V, since the $\mathbf{v_i}$'s span V, we can write $\mathbf{x} = c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}$ for some constants c_i. Now note that

$$T(\mathbf{x}) = T(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}) = c_1T(\mathbf{v_1}) + c_2T(\mathbf{v_2}) + \ldots + c_nT(\mathbf{v_n})$$
$$= c_1T'(\mathbf{v_1}) + c_2T'(\mathbf{v_2}) + \ldots + c_nT'(\mathbf{v_n}) = T'(c_1\mathbf{v_1} + c_2\mathbf{v_2} + \ldots + c_n\mathbf{v_n}) = T'(\mathbf{x})$$

so T and T' agree on every $\mathbf{x} \in V$, and thus they are the same function.

2. Let V be a vector space. I am telling you that, K, the set of *all functions* $f : V \to \mathbb{R}$ is a vector space (you don't need to prove it). Let V^* be the set of all linear transformations $T : V \to \mathbb{R}$. Prove that V^* is a vector space. (6 points)

V^* is a subset of K, which is a vector space. So it is sufficient to prove that V^* is a subspace. Note that the vectors of V^* are themselves linear transformations $T : V \to \mathbb{R}$. First we prove that the zero element of K is in V^*, ie $\mathbf{0}_K \in V^*$. The zero element of K is the function $f : V \to \mathbb{R}$ such that $f(\mathbf{v}) = 0$. This is easily seen to be a linear transformation, so it's in V^*.

We want to show that V^* is closed under addition. So we choose two elements of V^*, which are linear transformation T_1 and T_2, and we want to show that $T_3(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v})$ is a *linear transformation* (and thus $T_3 = T_1 + T_2 \in V^*$). We have to check two things. First note that

$$T_3(\mathbf{u}+\mathbf{v}) = T_1(\mathbf{u}+\mathbf{v}) + T_2(\mathbf{u}+\mathbf{v}) = T_1(\mathbf{u}) + T_1(\mathbf{v}) + T_2(\mathbf{u}) + T_2(\mathbf{v}) = (T_1(\mathbf{u}) + T_2(\mathbf{u})) + (T_1(\mathbf{v}) + T_2(\mathbf{v})) = T_3(\mathbf{u}) + T_3(\mathbf{v}).$$

So T_3 satisfies one condition to be a linear transformation. On the other hand

$$T_3(c\mathbf{u}) = T_1(c\mathbf{u}) + T_2(c\mathbf{u}) = c(T_1(\mathbf{u}) + T_2(\mathbf{u})) = cT_3(\mathbf{u}).$$

And thus $T_3 = T_1 + T_2$ is a linear transformation (and thus in V^*). Thus V^* is closed under addition.

To complete the proof we show that V^* is closed under scalar multiplication. So choose $T_1 \in V^*$ (T_1 is assumed to be a linear transformation) and $d \in \mathbb{R}$ and let $T_4(\mathbf{v}) = dT_1(\mathbf{v})$. We need to show that $T_4 = dT_1$ is in V^* (ie, that it is a linear transformation). We check two things. First note

$$T_4(\mathbf{u}+\mathbf{v}) = dT_1(\mathbf{u}+\mathbf{v}) = dT_1(\mathbf{u}) + dT_1(\mathbf{v}) = T_4(\mathbf{u}) + T_4(\mathbf{v}),$$

and so T_4 satisfies one condition to be a linear transformation. On the other hand

$$T_4(c\mathbf{u}) = dT_1(c\mathbf{u}) = dcT_1(\mathbf{u}) = c(dT_1(\mathbf{u})) = cT_4(\mathbf{u}).$$

Thus we see that T_4 is a linear transformation and thus $T_4 \in V^*$. *I would have accepted a less detailed explanation for full credit, but for the write-up, I decided to write EVERYTHING down.*

3. Let V be the set of all differentiable functions, $f(x)$, where $f : \mathbb{R} \to \mathbb{R}$. Define a function $T : V \to \mathbb{R}^2$ where $T(f(x)) = \begin{bmatrix} f(3) \\ f'(3) \end{bmatrix}$.

(a) Prove that T is a linear transformation. (5 points)

$$T(g(x) + h(x)) = \begin{bmatrix} g(3) + h(3) \\ g'(3) + h'(3) \end{bmatrix} = \begin{bmatrix} g(3) \\ g'(3) \end{bmatrix} + \begin{bmatrix} h(3) \\ h'(3) \end{bmatrix} = T(g(x)) + T(h(x))$$

and

$$T(cg(x)) = \begin{bmatrix} cg(3) \\ cg'(3) \end{bmatrix} = c\begin{bmatrix} g(3) \\ g'(3) \end{bmatrix} = cT(g(x))$$

so we are done!

(b) Is $f(x) = x^3 - 9x$ in the kernel of T? Why or why not. (2 points)

Note $f(3) = 27 - 27 = 0$ and $f'(3) = 3(3)^2 - 9 = 18$. So $T(f(x)) = \begin{bmatrix} 0 \\ 18 \end{bmatrix} \neq \mathbf{0}$. Thus $f(x)$ is *NOT* in the kernel!