MATH 217-4, EXTRA CREDIT #4

The purpose of this extra credit assignment is to prove that every vector space has a basis (possibly with infinite elements). We will end up using a tool called Zorn's lemma to do this. Zorn's lemma is basically an "axiom". This means that mathematicians simply assume it without proof. However, much of modern mathematics is based on it.

Before we continue, let us recall what it means for a subset $S \subset V$ to be a basis (S is just a possibly infinite number of "vectors" of V).

Definition 0.1. A subset $S \subset V$ is said to *span* V if every $\mathbf{v} \in V$ can be written as a linear combination of finitely many different elements of S. Explicitly this means that given $\mathbf{v} \in V$, then there are elements $\mathbf{s_1}, \mathbf{s_2}, \dots, \mathbf{s_q} \in S$ and scalars $r_1, \dots, r_q$ such that

$$r_1\mathbf{s_1} + r_2\mathbf{s_2} + \dots + r_q\mathbf{s_q} = \mathbf{v}.$$

Note that if you choose a different $\mathbf{v}' \in V$, you might need a different set of (finitely many elements of) S in order to write $\mathbf{v}'$ as a linear combination. In particular, you might need more than q of them, or you might be able to get away with less.

Definition 0.2. A subset $S \subset V$ is said to be *linearly independent* if for every finite set of elements of S, say $\{\mathbf{s_1}, \mathbf{s_2}, \dots, \mathbf{s_p}\} \subseteq S$, then the only way that

$$r_1\mathbf{s_1} + r_2\mathbf{s_2} + \dots + r_p\mathbf{s_p} = \mathbf{0}$$

is if $r_1 = 0, r_2 = 0, \dots, r_n = 0$. In other words, all of the r_i are necessarily zero. Note that you need to check this for every finite subset of elements, in particular you need to check *all* the sets, including those where $p = 5, 10, 100, 1000, \text{etc.}$

Definition 0.3. A subset $S \subset V$ is said to be a *basis for* V if it both spans V and is linearly independent.

We want to show that *EVERY* vector space has some basis. In particular, even vector spaces of continuous functions $\mathbb{R} \to \mathbb{R}$, or even vector spaces of kumquats (if we can figure out how to add them) always have a basis. (Note that these bases aren't going to be unique, just like bases of $\mathbb{R}^n$ aren't unique). Now we need to explain Zorn's lemma (at least in a special case). We begin with some terminology.

Let $\mathscr{P}$ be a collection of subsets of a set A. A specific collection $\mathscr{C}$ of the elements of $\mathscr{P}$ (which are themselves subsets of V, just like every element of $\mathscr{P}$ is) is called a chain if

- For every two elements $U, V \in \mathscr{P}$ either $U \subseteq V$ or $V \subseteq U$.

Before actually stating the lemma, let's do some examples of chains of subsets and chains.

Example 0.4. Start with $A = \mathbb{Z}$ (A is the integers). Let $\mathscr{P}$ be the set whose elements are all finite collections of integers. Note $\{1, 2, 3\}$ is an "element" of $\mathscr{P}$, as is $\{1, 61, -51234, 1278934, 0\}$. (We don't care about the order we write the integers down, so that $\{1, 2, 3\} = \{3, 2, 1\}$). We want to create a chain of these subsets.

- Note that neither $\{1, 2, 3\}$ or $\{1, 61, -51234, 1278934, 0\}$ is contained in the other

So lets consider the elements $\{\{1\},\{1,2\},\{1,2,3\}\{1,2,3,4\},\ldots,\{1,2,3,4,5,\ldots,n\},\ldots\}$. Note that this collection has an infinite number of different elements. However, given any two of them

$$\{1,2,\ldots,p\} \text{ and } \{1,2\ldots,q\}$$

one of those two is contained in the other (depending on if p is less than q).

Exercise 0.5. Let A and $\mathscr{P}$ be as above, find a different infinite "chain" of subsets. (1 point)

Example 0.6. Now we let $A=\mathbb{R}$ and we let $\mathscr{P}$ be the collection of all "open" intervals (a,b) where a and b are real numbers. For example, $(1,6)$, $(-6.4,\pi)$ and $(\sqrt{(13)},20)$ would all be "elements" of $\mathscr{P}$ (even though each one is also a subset of A). Consider the following chain of subsets $\mathscr{C}=\{(-a,a)$ where a is a positive real number$\}$. Note that give any two such elements of $\mathscr{C}$, one of them strictly contains the other!

Exercise 0.7. Let A and $\mathscr{P}$ be as above, find a different infinite "chain" of subsets. (1 point)

Lemma 0.8 (Zorn's Lemma)**.** *Let $\mathscr{P}$ be a collection of subsets of a bigger set A. Suppose that for every chain $\mathscr{C}$ of elements of $\mathscr{P}$ (which are themselves subsets of A) we automatically have $\bigcup_{C\in\mathscr{C}} C$ is also a subset of $\mathscr{P}$. In this case then $\mathscr{P}$ has at least one maximal sized subset.*

A couple comments. First, the notation $\bigcup_{C\in\mathscr{C}} C$ simply means we take all the elements in all the sets in the chain $\mathscr{C}$, and combine them together to get a new subset of A (which may or may not be in $\mathscr{P}$, hopefully it is). Second, we say that an element $M\in\mathscr{P}$ is maximal if no other element $N\in\mathscr{P}$ is strictly bigger than it (that is, you can't have $M\subsetneq N$)

Let's give a simple example.

Example 0.9. Let A be the integers, let $\mathscr{P}$ be the collection of finite subsets of A, and let $\mathscr{C}$ be the following chain of elements of $\mathscr{P}$.

$$\{\{1\},\{1,2\},\{1,2,3\},\{1,2,3,4\},\ldots,\{1,2,3,4,5,\ldots,n\},\ldots\}$$

This collection would *not* satisfy the conditions of Zorn's lemma because if you took the chain $\mathscr{C}$ and combined all those sets together, while you would get a subset of A (in fact you'd get all of A), this union is *not* equal to any of the subsets in $\mathscr{P}$ (which only have finitely many elements). If you made $\mathscr{P}$ bigger, so that you included A itself as an element of $\mathscr{P}$, then it would satisfy the condition of Zorn's lemma.

Exercise 0.10. If in the previous example, we made $\mathscr{P}$ bigger by including A into it. Find a maximal element of the new $\mathscr{P}$ (whose existence is guaranteed by Zorn's lemma). (1 point)

Now we move back to the world of vector spaces. Let V be a vector space and let $\mathscr{P}$ be the collection of all linearly independent subsets of V. We want to first show that Zorn's lemma applies in this situation, and so

Exercise 0.11. Let $\mathscr{C}$ be a chain of elements of $\mathscr{P}$. That is $\mathscr{C}$ is a bunch of linearly independent sets such that for every two of them, one is bigger than the other. Prove that $K=\bigcup_{C\in\mathscr{C}} C$ is itself linearly independent. (3 points) Hint: If you choose a finite collection of elements of K (which are actually "vectors" now), show that they must all be contained in a single one of the C's in the chain $\mathscr{C}$.

Ok, now we need just one more thing. Prove the following statement.

Exercise 0.12. Suppose that V is a vector space, L is a linearly independent set (possibly infinite) and $\mathbf{w}$ is an element of V that is *not* in the span of L. Prove that $\{\mathbf{w}\} \cup L$ is a linearly independent set. (2 points) Hint: There are two types of finite subsets of $\{\mathbf{w}\} \cup L$, those that contain w, and those that don't. Use this and the definition of linearly independent.

Now we are ready for you to prove that every vector space has a basis.

Exercise 0.13. Prove that every vector space has a basis. (2 points) Hint: Apply Zorn's lemma to $\mathscr{P} = \{$ all linearly independent subsets of $\}$ to obtain a maximal linearly independent set L (using 0.11). Now use 0.12 to prove that L must span V.