EXTRA CREDIT #1

MATH 217 – SECTION 4

The following will introduce you to the language of partitions of sets, and how they relate to what we've been doing in class recently. Later in the quarter, around chapter 4, we will build on this extra credit problem. Each question on this assignment is worth up to 2 points added to your homework score.

We begin with a definition of a partition. Let S be a set (such as the real numbers $\mathbb{R}$, real n-space $\mathbb{R}^n$, the set of integers $\mathbb{Z}$, or even just the numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$).

Definition 0.1. A *partition* of S is a collection of subsets $\{S_i\}$ of S, satisfying both of the following properties.

(i) Every element $x \in S$ (read x in S) is an element of some S_i.
(ii) No two distinct S_i and S_j have any elements in common.

Remark 0.2. The various S_i that make up a partition are called *partition subsets.*

We now do several examples.

If S is the set of integer $\mathbb{Z}$, then one partition of S is to make S_1 be the even integers and S_2 be the odd integers. Every integer is either even or odd (so every integer is either in S_1 or S_2) and thus condition (i) is satisfied. Likewise, no integer can be simultaneously both even and both odd (so no integer is in both S_1 and S_2) and thus condition (ii) is satisfied.

If S is the set of integers $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ then one possible partition is

- $S_1 = \{1\}$
- $S_2 = \{2, 3\}$
- $S_3 = \{4\}$
- $S_4 = \{5, 6, 7\}$
- $S_5 = \{8, 9\}$
- $S_6 = \{10\}$

obviously there are many other ways to "partition" this set of numbers as well.

Some partitions can have infinitely many subsets. For example if $S = \mathbb{R}^2$, the one can consider all the subsets

$$S_\lambda = \{\begin{bmatrix} x \\ \lambda \end{bmatrix} \in \mathbb{R}^2\} = \{\text{all pairs of numbers in } \mathbb{R}^2 \text{ where the second entry is } \lambda\}$$

Every element in $\mathbb{R}^2$ has some second entry, so every element in $\mathbb{R}^2$ is in S_λ for some λ. This takes care of partition condition (i). On the other hand, S_λ and S_δ have no terms in common if λ and δ are different, this takes care of partition condition (ii). Note in this example, every partition subset has infinitely many elements *and* there are infinitely many distrinct partition subsets. But it's still a partition.

Exercise 0.3. Find a partition of $S = \mathbb{Z}$ that has 4 partition subsets S_1, S_2, S_3 and S_4, such that each S_i has infinitely many elements.

An easy way to create partitions is to start with a function and partition the domain by the various solution sets (sometimes this is also called the level sets).

Proposition 0.4. *If $f : \mathbb{R}^n \to \mathbb{R}^m$ is a function (not necessarily a linear transformation, but those work too), then for each $\mathbf{b} \in \mathbb{R}^m$ (read $\mathbf{b}$ in $\mathbb{R}^m$) such that $f(\mathbf{x}) = \mathbf{b}$ has a solution, we consider the set of all solutions $\mathbf{x}$ to the equation $f(\mathbf{x}) = \mathbf{b}$. This is a subset of the domain, $\mathbb{R}^n$, that we will denote by $S_{\mathbf{b}}$. The collection of all such $S_{\mathbf{b}}$ forms a partition of the domain $\mathbb{R}^n$.*

Before trying the proof, lets do an example. Specifically, let $f : \mathbb{R} \to \mathbb{R}$ be the function that sends x to x^2 (that is, $f(x) = x^2$). This function gives us a partition on the domain, $\mathbb{R}$. The sets of the partition are S_λ for real numbers λ where $\lambda \geq 0$. We need λ to be bigger than zero since the square of a real number is never negative. Let us describe some of these sets,

- $S_0 = \{0\}$, the only element that becomes zero after squaring is zero.
- $S_1 = \{-1, 1\}$, both -1 and 1 become 1 after squaring.
- $S_4 = \{-2, 2\}$, both -2 and 2 become 4 after squaring.
- $S_a = \{-\sqrt{a}, \sqrt{a}\}$, both $-\sqrt{a}$ and $\sqrt{a}$ become a after squaring.

There are obviously an infinite number of distinct sets in this partition.

Let's let you do another example.

Exercise 0.5. Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined as the matrix transformation

$$T(\mathbf{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x}$$

Explicitly describe the partition of $\mathbb{R}^2$ associated to this function.

Exercise 0.6. Prove "Proposition 0.4". Hint: Remember, there are two properties to check, make sure to take care of both of them.

When working with linear (matrix) transformations, the partition of the domain associated to the matrix transformation is quite well behaved. In particular, every subset that forms the partition looks like every other (just translated around a bit). Let's make this precise.

Theorem 0.7. *Suppose that $T(\mathbf{x}) = A\mathbf{x}$ is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. Consider the partition subset $S_{\mathbf{0}}$ (that is, all solutions of the equation $T(\mathbf{x}) = \mathbf{0}$). Every other partition subset $S_{\mathbf{b}}$ is equal to $\mathbf{p} + S_{\mathbf{0}}$ for some choice of vector $\mathbf{p} \in \mathbb{R}^n$. Another way to say this is that $S_{\mathbf{b}}$ is just $S_{\mathbf{0}}$ translated by some vector $\mathbf{p} \in \mathbb{R}^n$.*

Remark 0.8. When I say the set $\mathbf{p} + S_{\mathbf{0}}$, I mean the set of all vectors that can be written as a sum $\mathbf{p} + \mathbf{v}$ where $\mathbf{v}$ is an element of $S_{\mathbf{0}}$.

The sets $\mathbf{p} + S_{\mathbf{0}}$ (for various choices of $\mathbf{p}$ that make up the partition of of $\mathbb{R}^n$) are sometimes called the cosets.

Exercise 0.9. Explain why $\mathbf{p}$ is always an element of the set $\mathbf{p} + S_{\mathbf{0}}$.

Exercise 0.10. Prove theorem 0.7.

Note that two different choices of $\mathbf{p}$ above can give the same particular partition subset. For example, in the exercise 0.5 above,

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} + S_{\mathbf{0}} = \begin{bmatrix} -7 \\ 3 \end{bmatrix} + S_{\mathbf{0}}$$

Exercise 0.11. Prove that $\mathbf{p} + S_{\mathbf{0}} = \mathbf{q} + S_{\mathbf{0}}$ if and only if $\mathbf{p} - \mathbf{q}$ is an element of $S_{\mathbf{0}}$.