1. Let A be the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$.

(a) Find a basis for Col$A$ and find a basis for Nul$A$. (12 points)

First we row reduce, and we get $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Clearly, the first column is the only pivot column, so a basis for Col$A$ is just $\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$.

To find a basis for Nul$A$, we row reduce the augmented matrix of usual homogeneous equation. After doing this, there is really only one equation to worry about $x_1+2x_2+3x_3 = 0$. So, $x_1 = -2x_2 - 3x_3$, and thus the parametric vector form is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$. Therefore $\{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}\}$ is a basis for Nul$A$.

(b) Show that $\begin{bmatrix} 12 \\ -3 \\ -2 \end{bmatrix}$ is in Nul$A$. Also write down it's coordinate vector in Nul$A$ with respect to the basis you found in part (a). (8 points)

First we compute $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ -3 \\ -2 \end{bmatrix}$ and note it is equal to zero, which proves the first part. For the second, note that $(-3)\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 12 \\ -3 \\ -2 \end{bmatrix}$, which means that $\begin{bmatrix} -3 \\ -2 \end{bmatrix}$ is the desired coordinate vector.

2. Let V be the collection of differentiable functions $f : \mathbb{R}^2 \to \mathbb{R}$ (that is, the partials of f with respect to x and y both exist). I am telling you that V *is a vector space*, that is, you may assume that V is a vector space without proving it.

Define a linear transformation $T : V \to \mathbb{R}^2$ by $T(f(x,y)) = \begin{bmatrix} \frac{\partial f}{\partial x}(5,-1) \\ \frac{\partial f}{\partial y}(5,-1) \end{bmatrix}$. Prove that T is a linear transformation. (15 points)

We need to show that T respects addition and scalar multiplication. Let $g(x), h(x)$ be elements of V and let c be a scalar in $\mathbb{R}$. First addition:

$$T(g(x)+h(x)) = \begin{bmatrix} \frac{\partial(g+h)}{\partial x}(5,-1) \\ \frac{\partial(g+h)}{\partial y}(5,-1) \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x}(5,-1) + \frac{\partial h}{\partial x}(5,-1) \\ \frac{\partial g}{\partial y}(5,-1) + \frac{\partial h}{\partial y}(5,-1) \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial x}(5,-1) \\ \frac{\partial g}{\partial y}(5,-1) \end{bmatrix} + \begin{bmatrix} \frac{\partial h}{\partial x}(5,-1) \\ \frac{\partial h}{\partial y}(5,-1) \end{bmatrix} = T(g(x)) + T(h(x)).$$

Next, scalar multiplication:

$$T(cg(x)) = \begin{bmatrix} \frac{\partial(cg)}{\partial x}(5,-1) \\ \frac{\partial(cg)}{\partial y}(5,-1) \end{bmatrix} = \begin{bmatrix} c\frac{\partial g}{\partial x}(5,-1) \\ c\frac{\partial(g)}{\partial y}(5,-1) \end{bmatrix} = c \begin{bmatrix} \frac{\partial g}{\partial x}(5,-1) \\ \frac{\partial(g)}{\partial y}(5,-1) \end{bmatrix} = cT(g(x)).$$

This proves that T is a linear transformation.

3. Let M be the 4×4 matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 1 & 3 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix}$$

(a) Compute $\det M$. (15 points)

First we do two row replacements to M to get

$$M' = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Note that this matrix is diagonal, and so has determinant $(1)(1)(-2)(1) = -2$. Since M' and M are only related by row replacements, $\det M = -2$ as well.

(b) What is the dimension of the Null space of M? Explain your reasoning. (5 points)

Since $\det M$ is non-zero, by the IMT, $\dim(\mathrm{Nul} M) = 0$.

4. Suppose that V and W are vector spaces and suppose that $T : V \to W$ is a surjective (aka, onto) linear transformation. If $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ is a spanning set for V, prove that $T(S) = \{T(\mathbf{v_1}), T(\mathbf{v_2}), \ldots, T(\mathbf{v_n})\}$ is a spanning set for W. (10 points)

We need to show, that for each $\mathbf{w} \in W$, there exists scalars $c_1, \ldots, c_n$ such that $\mathbf{w} = c_1 T(\mathbf{v_1}) + c_2 T(\mathbf{v_2}) + \ldots + c_n T(\mathbf{v_n})$. So choose an arbitrary $\mathbf{w} \in W$. Since T is surjective, there exists a $\mathbf{v} \in V$ such that $T(\mathbf{v}) = \mathbf{w}$. Since S spans V, there exist scalars, $c_1, \ldots, c_n$ such that $c_1\mathbf{v_1} + \ldots + c_n\mathbf{v_n} = \mathbf{v}$. Applying T gives us

$$c_1 T(\mathbf{v_1}) + \ldots + c_n T(\mathbf{v_n}) = T(c_1\mathbf{v_1} + \ldots + c_n\mathbf{v_n}) = T(\mathbf{v}) = \mathbf{w},$$

which completes our proof.

5. Suppose that V is a vector space and that H and K are subspaces of V. Prove that $H \cap K$ is also a subspace of V.

Reminder: Here $H \cap K$ is the set of all vectors $\mathbf{v} \in V$ such that $\mathbf{v}$ is in H *AND* $\mathbf{v}$ is in K. That is, $H \cap K$ is the vectors that are common to both H and K. (15 points)

We need to show that $H \cap K$ satisfies the three properties of a subspace.

- Zero: First we show that $\mathbf{0}$ is in $H \cap K$. But note that $\mathbf{0} \in H$ since H is a subspace, and $\mathbf{0} \in K$ since K is a subspace, so $\mathbf{0} \in H \cap K$ as desired.
- Addition: Choose $\mathbf{u}, \mathbf{v} \in H \cap K$, we want to show that their sum is in $H \cap K$. This implies that $\mathbf{u}, \mathbf{v} \in H$ so that $\mathbf{u}+\mathbf{v} \in H$ since H is a subspace. Likewise, this implies that $\mathbf{u}, \mathbf{v} \in K$ so that $\mathbf{u}+\mathbf{v} \in K$ since K is a subspace. Therefore $\mathbf{u}+\mathbf{v} \in H \cap K$ as desired.
- Scalars: Choose $\mathbf{u} \in H \cap K$ and choose c a real number, we want to show that $c\mathbf{u} \in H \cap K$. Since $\mathbf{u} \in H$, $c\mathbf{u} \in H$ because H is a subspace. Likewise, since $\mathbf{u} \in K$, $c\mathbf{u} \in K$ because K is a subspace. This proves that $c\mathbf{u} \in H \cap K$ as desired.

Therefore, $H \cap K$ is a subspace.

6. Let P_4 be the vector space of polynomials of degree less than or equal to 4. Consider the linear transformation $T : P_4 \to \mathbb{R}^3$ defined by

$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \\ p'(0) \end{bmatrix}$$

You do *NOT* need to prove that T is a linear transformation, you may assume that it is.

(a) Let $p(x) = x + 1$, $q(x) = x^2$, $r(x) = x$. Prove that $\{T(p(x)), T(q(x)), T(r(x))\}$ is a linearly independent set. (5 points)

Note $T(p(x)) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $T(q(x)) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $T(p(x)) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. There are many ways to show that these are linearly independent, here is one way. Consider the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. By doing a row replacement, we see that this matrix is row equivalent to $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, which obviously has non-zero determinant ($\det = 1 \neq 0$).

(b) Compute the dimension of the range of T. (5 points)

First note that the range of T is a subspace of $\mathbb{R}^3$. Since the range of T has 3 linearly independent vectors (by part (a)), $\dim(\text{range}T) \geq 3$. On the other hand, since it's a subspace of $\mathbb{R}^3$ (which has dimension 3), $\dim(\text{range}T) \leq 3$ (basically because you can't have more than three independent vectors in $\mathbb{R}^3$). Therefore, the only possibility is that $\dim(\text{range}T) = 3$.

(c) Compute the dimension of the kernel of T. (5 points)

By the rank theorem, $\dim(\text{range}T) + \dim(\ker T) = \dim P_4$. But $\dim P_4 = 5$, and so using part (b), we see that $\dim(\ker T) = 2$.

(EC) Let V be a vector space and let K be the set of functions $f : V \to \mathbb{R}$. K is a vector space, you don't need to prove it.

For solutions to the extra credit, see the already posted solutions to Quiz #7.

(a) Let $z : V \to \mathbb{R}$ be the element of K such that $z(\mathbf{v}) = 0$ for every $\mathbf{v} \in V$. Prove that z is a linear transformation. (1 point)

(b) Let T_1 and T_2 be elements of K that are also linear transformations. Prove that T_3 is a linear transformation where $T_3(\mathbf{v}) = T_1(\mathbf{v}) + T_2(\mathbf{v})$ for any $\mathbf{v} \in V$. (1 point)

(c) Let T_1 be an element of K that is also a linear transformation. If d is any element of $\mathbb{R}$, prove that T_4 is a linear transformation where $T_4(\mathbf{v}) = dT_1(\mathbf{v})$. (1 point)

(d) Let V^* be the set of all linear transformations $T : V \to \mathbb{R}$. Prove that V^* is a vector space. (2 points)