1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation such that

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0.5 \\ 2 \end{bmatrix}, \text{ and } T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

(a) Write down a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ (10 points).

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 3 & 0.5 & 4 \\ 0 & 2 & 3 \end{bmatrix}$$

(b) Find an inverse to A or say why it doesn't exist. If you can't figure out part (a), use the matrix $\begin{bmatrix} 1 & 1 & 0 \\ 3 & 4 & \frac{1}{3} \\ 0 & 8 & 4 \end{bmatrix}$. (20 points)

$$A^{-1} = \begin{bmatrix} 13 & -4 & 1 \\ 18 & -6 & 2 \\ -12 & 4 & -1 \end{bmatrix}$$

(c) *Explain* why the product $A^{-1}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ should equal $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Check if your inverse matrix satisfies this property. (10 points)

We have, $A^{-1}\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = A^{-1}A\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = I_{3\times 3}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. In our particular case, we just note that

$$(1)(13) + (3)(-4) + (0)(1) = 1,$$
$$(1)(18) + (-6)(3) + (0)(2) = 0 \text{ and}$$
$$(1)(-12) + (3)(4) + (0)(-1) = 0.$$

2. Suppose M is a 3×4 matrix. If the system of equations corresponding to $M\mathbf{x} = \mathbf{0}$ has two free variables, is it possible that the linear transformation

$$\mathbf{x} \to M\mathbf{x}$$

is surjective (recall surjective means onto) or injective (recall injective means one-to-one)? Why or why not? (15 points)

Since M has two free variables, it has two columns without pivots. Since it has four columns, it must have only 2 pivots. Therefore we note that M doesn't have a pivot in every column, and so the associated linear transform is *NOT* injective. We likewise note that M doesn't have a pivot in every row, and so we know that the associated linear is *NOT* surjective.

3. Suppose that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is a linearly independent set of vectors in $\mathbb{R}^6$. Further suppose that $\mathbf{w}$ is a vector in $\mathbb{R}^6$ such that

$\mathbf{w}$ is *not* an element of $\text{Span}(\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3})$.

Is it true that $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{w}\}$ is linearly independent? Why or why not? (15 points)

It is linearly independent. Suppose that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} + c_4\mathbf{w} = 0$ for some constants c_i. We want to show that these constants *must* all be zero (which will show what we want). We consider two cases.

Case 1: $c_4 \neq 0$. But then we can write $\mathbf{w} = \frac{-c_1}{c_4}\mathbf{v_1} + \frac{-c_2}{c_4}\mathbf{v_2} + \frac{-c_3}{c_4}\mathbf{v_3}$ which is impossible since $\mathbf{w}$ is *not* in the span of $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$.

Case 2: $c_4 = 0$. Then we have that $c_1\mathbf{v_1} + c_2\mathbf{v_2} + c_3\mathbf{v_3} = 0$ which implies that c_1, c_2, and c_3 must be zero as well.

Therefore, the only possibility is that all the c_i are zero which proves what we wanted.

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the linear transformation associated to the matrix

$$M = \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & -1 \end{bmatrix}.$$

Write out the solution to $T(\mathbf{x}) = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ in parametric vector form. (15 points)

The reduced echelon form of the associated augmented matrix is $\begin{bmatrix} 1 & 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Writing out our equations we get that

$$x_1 + x_3 + x_4 = 3 \text{ and } x_2 + x_3 - x_4 = 1 \text{ with } x_3 \text{ and } x_4 \text{ free.}$$

Therefore the parametric form of this solution set is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

5. Suppose that A and B are 22×22 matrices. Suppose that A is invertible but AB is *not* invertible. Prove that the linear transformation $T(\mathbf{x}) = B\mathbf{x}$ is *not* injective (which is to say, is not one-to-one). (15 points)

It is enough to show that $T(\mathbf{x}) = \mathbf{0}$ has a non-trivial solution, and so that is what we will do. Since AB is not invertible (and it is square), $(AB)\mathbf{x} = \mathbf{0}$ has a nontrivial solution. So

$$A^{-1}(AB)\mathbf{x} = A^{-1}\mathbf{0} = \mathbf{0}$$

has a non-trivial solution as well. However, $A^{-1}(AB)\mathbf{x} = IB\mathbf{x} = B\mathbf{x}$. Therefore, the equation $B\mathbf{x} = \mathbf{0}$ has a nontrivial solution and we are done.

(EC) Suppose that A is an invertible 6×6 matrices and B is an invertible 4×4 matrix. If

$$C = \begin{bmatrix} 0_{6\times 4} & A \\ B & 0_{4\times 6} \end{bmatrix},$$

then find C^{-1} as a partitioned matrix in terms of A^{-1} and B^{-1}. Explicitly state the sizes of the various pieces of the partitioned C^{-1}. Here $0_{6\times 4}$ means the matrix with all entries zero and with 6 rows and 4 columns. (5 points)

First we check that the matrix C really is square. Since A has 6 columns, we see that C has $6 + 4 = 10$ columns. Since B has 4 rows, we see that C has $4 + 6 = 10$ rows. Therefore C is square. So break up C^{-1} into

$$D = C^{-1} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}$$

where D_{11} is 4×6, D_{12} is 4×4, D_{21} is 6×6 and D_{22} is 6×4. We know that

$$CD = I_{10\times 10} = \begin{bmatrix} I_{6\times 6} & 0_{6\times 4} \\ 0_{4\times 6} & I_{4\times 4} \end{bmatrix}$$

Therefore we have the following four equations,

$$0_{6\times 4}D_{11} + AD_{21} = I_{6\times 6} \tag{1}$$

$$BD_{11} + 0_{4\times 6}D_{21} = 0_{4\times 6} \tag{2}$$

$$0_{6\times 4}D_{12} + AD_{22} = 0_{6\times 4} \tag{3}$$

$$BD_{12} + 0_{4\times 6}D_{22} = I_{4\times 4} \tag{4}$$

We first consider equation (2). This is just $BD_{11} = 0_{4\times 6}$. Since B is invertible, we see that $D_{11} = B^{-1}0_{4\times 6} = 0_{4\times 6}$. Likewise when considering equation (3), we see that $AD_{22} = 0_{6\times 4}$, and therefore, since A is invertible, $D_{22} = A^{-1}0_{6\times 4} = 0_{6\times 4}$. Equation (1) says that $AD_{21} = I_{6\times 6}$ and so $D_{21} = A^{-1}$ since D_{21} is square. Likewise, equation (4) says that $BD_{12} = I_{4\times 4}$ and so $D_{12} = B^{-1}$ since D_{12} is square.

In summary,

$$D = \begin{bmatrix} 0_{4\times 6} & B^{-1} \\ A^{-1} & 0_{6\times 4} \end{bmatrix}.$$

A quick check verifies that D is indeed is the inverse to C.