MATH 185

DUE WEDNESDAY, NOVEMBER 18TH

Recall the following definition:

Definition 1.1. Given a function T (which takes vectors as input, and outputs vectors), we say that T is a *linear transformation* if the following two properties hold.

(#1) For any two vectors $\mathbf{v}$ and $\mathbf{v}'$, we always have $T(\mathbf{v}+\mathbf{v}') = T(\mathbf{v}) + T(\mathbf{v}')$.
(#2) For any $\mathbf{v}$ and any real number c, we have $T(c\mathbf{v}) = cT(\mathbf{v})$.

Exercise 1.2. Prove that if T is a linear transformation, then $T(\mathbf{0}) = \mathbf{0}$.

Note $T(0(\mathbf{0})) = 0T(\mathbf{0}) = 0$.

We also need the following definition.

Definition 1.3. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation (for now, $n = 2$ or $n = 3$, but both n's are the same). We say that T is *invertible* if there exists a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$T \circ S = \mathrm{id}_{\mathbb{R}^n}$$

and

$$S \circ T = \mathrm{id}_{\mathbb{R}^n}\,.$$

In this case we say that S is the *inverse of* T.

Example 1.4. Consider the linear transformation T that take a vector and reverses it's direction, but leaves the magnitude alone (you can verify that it is indeed a linear transformation since $T(\mathbf{v}) = -\mathbf{v}$). Then T is invertible since you can set $S = T$ and then $T \circ S = T \circ T = S \circ T = \mathrm{id}_{\mathbb{R}^n}$ (ie, $-(-\mathbf{v}) = \mathbf{v}$).

Example 1.5. Consider the linear transformation T that takes a vector $\mathbf{u}$ and doubles it's length (ie $T(\mathbf{u}) = 2\mathbf{u}$). Then T is invertible and it's inverse is S which takes a vector and halves it's length (ie $S(\mathbf{u}) = \frac{1}{2}\mathbf{u}$).

Example 1.6. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that is the constant function $T(\mathbf{u}) = \mathbf{0}$. Then T is not invertible. (We'll see why in a moment).

Exercise 1.7. Suppose that T is invertible, prove that T is surjective.
Hint: Write $T \circ S = \mathrm{id}_{\mathbb{R}^n}$.

Fix a vector $\mathbf{y}$. Set $\mathbf{x} = S(\mathbf{y})$. Then $T(\mathbf{x}) = T(S(\mathbf{y})) = \mathbf{y}$ so T is surjective.

Exercise 1.8. Suppose that T is invertible, prove that T is injective.
Hint: Write $S \circ T = \mathrm{id}_{\mathbb{R}^n}$.

Suppose that $T(\mathbf{u}) = T(\mathbf{v})$. Then $\mathbf{u} = S(T(\mathbf{u})) = S(T(\mathbf{v})) = \mathbf{v}$ and T is injective.

Exercise 1.9. Suppose that T is a bijective linear transformation (ie, injective and surjective). Prove that T is invertible.
Hint: It's not so hard to figure out what the definition of S has to be. But then you have to verify that S is also a linear transformation

For each vector $\mathbf{x}$, define $S(\mathbf{x})$ to be the unique vector $\mathbf{y}$ such that $T(\mathbf{y}) = \mathbf{x}$. We see that $T(S(\mathbf{x})) = T(\mathbf{y}) = \mathbf{x}$ (for any $\mathbf{x}$) and also that $S(T(\mathbf{y})) = S(\mathbf{x}) = \mathbf{y}$ (for any $\mathbf{y}$). We will now show that S is a linear transformation.

Choose vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ and write $\mathbf{x}_1 = T(\mathbf{y}_1)$ and $x_2 = T(\mathbf{y}_2)$ (we can do this because T is surjective). Then

$$S(\mathbf{x}_1 + \mathbf{x}_2) = S(T(\mathbf{y}_1) + T(\mathbf{y}_2)) = S(T(\mathbf{y}_1 + \mathbf{y}_2)) = \mathbf{y}_1 + \mathbf{y}_2 = S(\mathbf{x}_1) + S(\mathbf{x}_2).$$

Likewise

$$S(c\mathbf{x}_1) = S(cT(\mathbf{y}_1)) = S(T(c\mathbf{y}_1)) = c\mathbf{y}_1 = cS(\mathbf{x}_1)$$

Exercise 1.10. Suppose that $T : \mathbb{R}^n \to \mathbb{R}^n$ is invertible ($n = 2$ or 3). Prove that the inverse of T is unique. (Suppose that S and S' are both inverses of T. Then prove that $S = S'$).

Suppose that S and S' are both inverses of T. Then $T(S(\mathbf{x})) = \mathbf{x} = T(S'(\mathbf{x}))$ for all $\mathbf{x}$. But then $S(T(S(\mathbf{x}))) = S(T(S'(\mathbf{x})))$ for all $\mathbf{x}$ so that $S(\mathbf{x}) = S'(\mathbf{x})$ for all $\mathbf{x}$. Thus $S = S'$.

Exercise 1.11. Show that $T : \mathbb{R}^n \to \mathbb{R}^n$ is injective ($n = 2$ or 3) if and only if T is surjective

Suppose first that T is injective and fix $\mathbf{y} \in \mathbb{R}^n$. Let $\mathbf{u}_1, \dots, \mathbf{u}_n$ be a basis. Then $T(\mathbf{u}_1), \dots, T(\mathbf{u}_n)$ is also a basis by the previous worksheet. Thus there exists $a_1, \dots, a_n$ such that $\mathbf{y} = a_1 T(\mathbf{u}_1) + \cdots + a_n T(\mathbf{u}_n) = T(a_1 \mathbf{u}_1 + \cdots + a_n \mathbf{u}_n)$ and so T is surjective.

Conversely, suppose that T is surjective. Set $\mathbf{x}_1, \dots, \mathbf{x}_n$ to be a basis for $\mathbb{R}^n$. Since T is surjective there exists elements $\mathbf{u}_1, \dots, \mathbf{u}_n$ such that $T(\mathbf{u}_i) = \mathbf{x}_i$ for all i. We will show that $\mathbf{u}_1, \dots, \mathbf{u}_n$ is linearly independent and thus a basis. Suppose that $a_1, \dots, a_n$ are scalars and that $a_1 \mathbf{u}_1 + \dots a_n \mathbf{u}_n = \mathbf{0}$. Then

$$\mathbf{0} = T(\mathbf{0}) = T(a_1 \mathbf{u}_1 + \dots a_n \mathbf{u}_n) = a_1 T(\mathbf{u}_1) + \cdots + a_n T(\mathbf{u}_n) = a_1 \mathbf{x}_1 + \cdots + a_n \mathbf{x}_n$$

and so $a_1 = 0, \dots, a_n = 0$ (since the $\mathbf{x}_i$ are linearly independent). But then $\mathbf{u}_1, \dots, \mathbf{u}_n$ is also a linearly independent set as claimed. But now T takes a basis (the $\mathbf{u}_i$) to another basis (the $\mathbf{x}_i$) and so T is injective by the previous worksheet.