WORKSHEET #2

In this worksheet, we'll learn about another way to think about continuity. First we need to define some terms.

1. Definitions

Recall that given a set T (of things, possibly T is a bunch of numbers), a subset U of T is a collection of things inside T. For example, $\{1,2,4,7\}$ is a subset of $\mathbb{N}$. The even numbers are also a subset of $\mathbb{N}$. Furthermore $\mathbb{N}$ is a subset of $\mathbb{Z}$ and $\mathbb{Z}$ is a subset of $\mathbb{Q}$. And finally $\mathbb{Q}$ is a subset of $\mathbb{R}$.

If U is a subset of T, we write $U \subseteq T$. So in the previous example, we have

$$\{1,2,4,7\} \subseteq \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

Given two subsets U and V of $\mathbb{R}$, we can construct two other subsets using them. The union of two sets U and V, denoted by $U \cup V$ is the collection of all elements of $\mathbb{R}$ that are in either U or V. The intersection of U and V, denoted by $U \cap V$ is all the elements of $\mathbb{R}$ that are in both of U and V. See section 0-4 in the book for additional discussion of these notions.

Hopefully the above was all pretty easy. We begin the real definitions with the notion of the *inverse image of a subset.*

Definition 1.1. Let $f : S \to \mathbb{R}$ be a function where S is the domain of f. Suppose that U is a subset of $\mathbb{R}$ (that is, $U \subseteq \mathbb{R}$). We define the *inverse image of U under f*, denoted by $f^{-1}(U)$ to be the following subset of S.

$$f^{-1}(U) = \{x \in S | f(x) \in U\}$$

In other words, $f^{-1}(U)$ is all the elements of S that f sends into U.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by the formula $f(x) = x^2$. Let U be the interval $(1,4)$. Then $f^{-1}(U)$ is made up of two intervals $(-2,-1)$ and $(1,2)$. We can use a union sign to represent $f^{-1}(U) = (-2,-1) \cup (1,2)$.

Exercise 1.3. Consider $g : (-4\pi, 4\pi) \to \mathbb{R}$ be defined by $g(x) = \sin(x)$. Suppose $U = (0,1)$. Compute $g^{-1}(U)$.

Proof. It is easy to see that $g^{-1}(U) = (-4\pi, -7\pi/2) \cup (-7\pi/2, -3\pi) \cup (-2\pi, -3\pi/2) \cup (-3\pi/2, -\pi) \cup (0, \pi/2) \cup (\pi/2, \pi) \cup (2\pi, 5\pi/2) \cup (5\pi/2, 3\pi)$. ∎

Finally we define the notion of an open set.

Definition 1.4. We say that a subset $U \subset \mathbb{R}$ is *open* if for every element $c \in U$, there exists a positive real number $d > 0$ such that $(c - d, c + d) \subseteq U$.

Exercise 1.5. Suppose that $U = (a, b)$ is a non-empty open interval. Prove that U is an open set.

Proof. Choose $c \in U$. Let $d = \min(b - c, c - a)$ and note $d > 0$. We wish to show that $(c-d, c+d) \subseteq U$. It is sufficient to show that $a \leq c-d$ and $b \geq c+d$. But by the definition of d, we know that $d \leq b - c$ and so rearranging gives us $b \geq c + d$ as desired. Likewise, $d \leq c - a$ and so $a \leq c - d$ and we are done. $\square$

Exercise 1.6. Give an example of an open subset of $\mathbb{R}$ that is not an open interval.

Proof. Let $W = (0, 1) \cup (1, 2)$. It is easy to see that U is not even an interval (let alone an open one), since $1 \notin U$ but 0.5 and 1.5 are in U. We need to now show that it is open. Instead of proving this directly, we will refer to the next exercise. Note that $U = (0, 1)$ and $V = (1, 2)$ are open by the previous exercise, so that $U \cup V$ is open assuming we do the next exercise correctly. $\square$

Exercise 1.7. Suppose that U and V are two open subsets of $\mathbb{R}$. Prove that $U \cup V$ is an open subset of $\mathbb{R}$. Also prove that $U \cap V$ is an open subset of $\mathbb{R}$.

Proof. First we do the situation of $U \cup V$. Choose $c \in U \cup V$. Then either $c \in U$ or $c \in V$. We do two cases. Case #1: if $c \in U$ then since U is open, there exists some $d > 0$ so that $(c-d, c+d) \subseteq U$. But $U \subseteq U \cup V$ so $(c-d, c+d) \subseteq U \subseteq U \cup V$ as desired. Case #2: if $c \in V$ then since V is open there exists a $e > 0$ such $(c-e, c+e) \subseteq V \subseteq U \cup V$ as desired.

Now we prove that $U \cap V$ is open. Choose $c \in U \cap V$. Then $c \in U$ *AND* $c \in V$. Since $c \in U$, there exists $d_1 > 0$ such that $(c-d_1, c+d_1) \subseteq U$. Likewise there exists a $d_2 > 0$ such that $(c-d_2, c+d_2) \subseteq V$. Let $d = \min(d_1, d_2)$. Note that $d \leq d_1$ and $d \leq d_2$ by construction. Then $(c-d, c+d) \subseteq (c-d_1, c+d_1)$ and $(c-d, c+d) \subseteq (c-d_2, c+d_2)$. But then

$$(c-d, c+d) \subseteq (c-d_1, c+d_1) \subseteq U$$

and

$$(c-d, c+d) \subseteq (c-d_2, c+d_2) \subseteq V.$$

So ever element of $(c-d, c+d)$ is in U, and every element is also in V. And so we conclude that $(c-d, c+d) \subseteq U \cap V$ as desired. $\square$

Exercise 1.8. Suppose that a set U is open. Prove that U is a (possibly infinite) union of open intervals.

Proof. For each $c \in U$, choose $d_c > 0$ such that $(c-d_c, c+d_c) \subseteq U$. Now consider

$$\bigcup_{c \in U} (c-d_c, c+d_c).$$

We will show that this union is equal to U. First note that $U \subseteq \bigcup_{c \in U}(c-d_c, c+d_c)$, because for every point $c_0 \in U$,

$$c_0 \in (c_0 - d_{c_0}, c_0 + d_{c_0}) \subseteq \bigcup_{c \in U} (c-d_c, c+d_c).$$

On the other hand, $\left(\bigcup_{c \in U}(c-d_c, c+d_c)\right) \subseteq U$ because every term in the union is contained in U. Thus $\square$

2. Another characterization of continuity

Our real goal for the day is the following theorem, that you will prove shortly.

Theorem 2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at every point c of $\mathbb{R}$ if and only if, for every open subset U of $\mathbb{R}$ (thought of as in the codomain), $f^{-1}(U)$ is an open subset of $\mathbb{R}$ (thought of as in the domain).*

Note that in this theorem, there is an "if and only if", which means both conditions are equivalent.

Exercise 2.2. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at every $c \in \mathbb{R}$, prove that for every open subset U of $\mathbb{R}$, $f^{-1}(U)$ is an open subset of $\mathbb{R}$.

Proof. Choose $c \in f^{-1}(U)$. Then $f(c) \in U$. Since U is open, there exists some $e > 0$ such that $(f(c) - e, f(c) + e) \subset U$. Then, since f is continuous at c (thinking of $\epsilon = e$) we know that there exists a $d = \delta$ such that for all $x \neq c$ satisfying $c - d < x < c + d$ (in other words, $x \in (c - d, c + d)$), we have $f(c) - e < f(x) < f(c) + e$ (in other words, $f(x) \in (f(c) - e, f(c) + e)$). But then

$$f(x) \in (f(c) - e, f(c) + e) \subseteq U.$$

In other words, $x \in f^{-1}(U)$. So since this works for every $x \in (c - d, c + d)$, we see that $(c - d, c + d) \subset f^{-1}(U)$ as desired (that is, we found our d). $\square$

Exercise 2.3. Suppose that for every open subset $U \subseteq \mathbb{R}$, we have $f^{-1}(U)$ is an open subset of $\mathbb{R}$. Prove that f is continuous.

Proof. Choose $c \in \mathbb{R}$ and pick $\epsilon > 0$. Consider the open interval $(f(c) - \epsilon, f(c) + \epsilon)$. By a previous exercise, this interval is an open set. Call this open set U. By hypothesis, $f^{-1}(U)$ is open, and since $c \in f^{-1}(U)$ (because $f(c) \in (f(c) - \epsilon, f(c) + \epsilon) = U$), we know there exists a $d > 0$ such that $(c - d, c + d) \subseteq f^{-1}(U)$. Set $\delta = d$. For any $x \neq c$ such that $c - d < x < c + d$, we see that $x \in (c - d, c + d)$. Therefore $x \in f^{-1}(U)$ and so $f(x) \in U = (f(c) - \epsilon, f(c) + \epsilon)$ which means that $f(c) - \epsilon < f(x) < f(c) + \epsilon$ as desired. $\square$