SOLUTION TO HOMEWORK #4, PROBLEM (5)

(5) Suppose that f is a function that is continuous at a and that $f(a) > 0$. Show that there exists an open interval (b, c) containing a such that $f(x) > 0$ for all $x \in (b, c)$.

Proof. Set $\varepsilon = f(a)/2$. Notice that $\varepsilon > 0$ because $f(a) > 0$. Because f is continuous, there exists a δ such that whenever x satisfies $|x - a| < \delta$, then we have that $|f(x) - f(a)| < \varepsilon = f(a)/2$. [1] Set $b = a - \delta$, $c = a + \delta$. Saying that $|x - a| < \delta$ is exactly the same thing as asserting that $x \in (b, c)$. Thus all we have to do is verify that for those x, $f(x) > 0$. But now $|f(x) - f(a)| < f(a)/2$ implies that $f(a) - f(x) < f(a)/2$. Re-arranging this last inequality gives us $f(x) > f(a)/2 > 0$ which is what we wanted. □

Remark 0.1. In the previous proof, you could have replaced the $\varepsilon = f(a)/2$ with $f(a)$, or $f(a)/7$ etc. But $\varepsilon = 2f(a)$ won't work! (Explain to yourself why that is).

You may ask how one comes up with the particular ε we used. To see this, draw yourself a picture. We know that $f(a) > 0$, so in this problem our goal is to force $f(x)$ above zero for x near a. For a continuous graph, this should be pretty obviously true, so what ϵ interval do you need to pick on the y-axis to make it true?

[1] Notice I didn't put the $0 <$ before the $|x - a|$, I can do that because we are assuming that f is continuous so that the case when $x = a$ is automatically satisfied.