NOTES ON CONNECTED AND DISCONNECTED SETS

In this worksheet, we'll learn about another way to think about continuity. First we need to define some terms.

Definition 0.1. A subset $K \subseteq [a, b]$ is called *an open subset of* $[a, b]$ if there exists an open set U of $\mathbb{R}$ such that $U \cap [a, b] = K$.

Proposition 0.2. *Suppose that $f : [a, b] \to \mathbb{R}$ is a function. Then f is continuous if and only if for every open subset U of $\mathbb{R}$, $f^{-1}(U)$ is an open subset of $[a, b]$.*

Remark 0.3. This proposition is kind of a pain to prove because there are so many special cases to write down. But hopefully no special case is very hard.

Proof. We have two directions to prove. First suppose that f is continuous. Let U be an open subset of $\mathbb{R}$ we need to show that $f^{-1}(U) = U' \cap [a, b]$ for some open U' (which will show that $f^{-1}(U)$ is an open subset of $[a, b]$). Set

$$U' = f^{-1}(U) \cup (a - 3, a) \cup (b, b + 2)$$

We need to show that U' is open and that $U' \cap [a, b] = f^{-1}(U)$. The second statement is easy because $U' \cap [a, b] = \{x \in \mathbb{R} | x \in f^{-1}(U)\}$ (the other parts of the set U' aren't in $[a, b]$). To show that U' is open choose $x_0 \in U'$. Since U is open, there exists $\epsilon > 0$ such that $(x - \epsilon, x + \epsilon) \subseteq U$. There are now five cases.

(1) $x_0 = a$. Since f is continuous at a, there exists $\delta > 0$ such that for $x \in [a, a + \delta)$, $f(x) \in (x-\epsilon, x+\epsilon) \subseteq U$ so that $x \in f^{-1}(U)$. Set $d = \min(\delta, 3)$. Then we immediately see that $(a - d, a + d) \subseteq (a - 3, a) \cup [a, a + \delta) \subseteq (a - 3, a) \cup f^{-1}(U) \subseteq U'$, as desired.
(2) $x_0 = b$. Since f is continuous at b, there exists $\delta > 0$ such that for $x \in (b - \delta, b]$, $f(x) \in (x-\epsilon, x+\epsilon) \subseteq U$ so that $x \in f^{-1}(U)$. Set $d = \min(\delta, 2)$. Then we immediately see that $(b - d, b + d) \subseteq (b - \delta, b] \cup (b, b + 2) \subseteq f^{-1}(U) \cup (b, b + 2) \subseteq U'$, as desired.
(3) $x_0 \in (a, b)$. In this case, there exists a $\delta > 0$ such that for all $x \in (x_0-\delta, x_0+\delta) \subseteq [a, b]$, we have that $f(x) \in (x - \epsilon, x + \epsilon) \subseteq U$. In particular, $(x_0 - \delta, x_0 + \delta) \subseteq f^{-1}(U) \subseteq U'$.
(4) $x_0 \in (a - 3, a)$. But $(a - 3, a)$ is an open set so we can find $d > 0$ such that $(x_0 - d, x_0 + d) \subseteq (a - 3, a) \subseteq U'$.
(5) $x_0 \in (b, b + 2)$. But $(b, b + 2)$ is an open set so we can find $d > 0$ such that $(x_0 - d, x_0 + d) \subseteq (b, b + 2) \subseteq U'$.

Therefore U' is open and we are done with the first direction.

Now suppose that f satisfies the property that for every open set $U \subseteq \mathbb{R}$, $f^{-1}(U)$ is an open subseteq of $[a, b]$. We need to show that f is continuous. Choose $c \in [a, b]$ and fix $\varepsilon > 0$. Observe that $U = (f(c) - \epsilon, f(c) + \epsilon)$ is open, and so is its inverse image by f, so that there exists an open set $U' \subseteq \mathbb{R}$ such that $U' \cap [a, b] = f^{-1}(U)$. We do three cases.

(1) $c \in (a, b)$. Thus $c \in (a, b) \cap U'$ which is also open. So there exists a $\delta = d > 0$ such that $(c - d, c + d) \subseteq (a, b) \cap U'$. For any $x \in (c - d, c + d)$, we have that $x \in f^{-1}(U) = U' \cap [a, b]$. Thus $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$ as desired.

(2) $c = a$. So there exists a $d > 0$ such that $(c-d, c+d) \subseteq U'$. Choose $\delta = \min(d, b-a)$. Then for $x \in [c, c+\delta) \subseteq [a,b] \cap U' = f^{-1}(U)$, we see that $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$ as desired.
(3) $c = b$. So there exists a $d > 0$ such that $(c-d, c+d) \subseteq U'$. Choose $\delta = \min(d, b-a)$. Then for $x \in (c-\delta, c] \subseteq [a,b] \cap U' = f^{-1}(U)$, we see that $f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$ as desired.

This completes the proof of the other direction. □

Definition 0.4. A set of real numbers A is called *disconnected* if there exist two open subsets of $\mathbb{R}$, call them U and V such that

(1) $A \cap U \cap V = \emptyset$.
(2) $A \subseteq U \cup V$
(3) $A \cap U \neq \emptyset$.
(4) $A \cap V \neq \emptyset$.

In such a case, we call U and V form a *disconnection of* A (or we simply say they disconnect A).

A set of real numbers A is called *connected* if it is not disconnected.

Example 0.5. The set $(0,1) \cup (1,2)$ is disconnected. Choose $U = (0,1)$ and $V = (1,2)$. $U \cap V = \emptyset$ so condition (1) is satisfied. $U \cup V = A$ so condition (2) is satisfied. We also have $A \cap U = (0,1) \neq \emptyset$, so condition (3) is satisfied. Finally we have that $A \cap V = (1,2)$ so condition (4) is satisfied.

Example 0.6. The set $\mathbb{Z}$ is disconnected. Choose $U = (-\infty, 0.5)$, $V = (0.5, \infty)$. I'll let you verify statements (1) through (4).

Example 0.7. Suppose that $a, b \in A$, and that $a < b$. Further suppose that $a < c < b$ but that $c \notin A$. Let $U = (-\infty, c)$, $V = (c, \infty)$.

Proposition 0.8. *Suppose that $f : [a,b] \to \mathbb{R}$ is continuous. Further suppose that D is a disconnected non-empty subset of the image of f (ie, V is disconnected, $D \subseteq \{$image of $f\}$). Then the set $f^{-1}(D)$ is disconnected.*

Proof. By assumption, there exists open sets U and V that disconnect D. Note that $f^{-1}(U)$ is an open subset of $[a,b]$ and also that $f^{-1}(V)$ is an open subset of $[a,b]$. Thus, there exist open sets U' and V' such that $U' \cap [a,b] = f^{-1}(U)$ and $V' \cap [a,b] = f^{-a}(V)$. I claim that U' and V' disconnect $f^{-1}(D)$. To show this we have to verify properties (1) through (4) from above.

To verify (1), suppose that $x \in f^{-1}(D) \cap U' \cap V'$, we will aim for a contradiction. Thus $f(x) \in D$. On the other hand, $x \in f^{-1}(D) \subseteq [a,b]$ so $x \in U' \cap [a,b] = f^{-1}(U)$. In particular, $f(x) \in U$. Likewise, $x \in V' \cap [a,b]$ so that $x \in f^{-1}(V)$. Therefore $f(x) \in V$. Thus $f(x) \in U \cap V \cap D = \emptyset$, a contradiction.

To verify (2), suppose that $x \in f^{-1}(D)$. We will show that $x \in U' \cup V'$. Note first that $f(x) \in D \subseteq U \cup V$. Thus either $f(x) \in U$ of $f(x) \in V$. In the first case, we see that $x \in f^{-1}(U) \subseteq U'$. In the second case we obtain that $x \in f^{-1}(V) \subseteq V'$. Thus in either case $x \in U' \cup V'$ and (2) is verified.

To verify (3), choose $y \in D \cap U$ (note such a y exists by hypothesis). Since $y \in D \subseteq$ image of f there exists $x \in [a,b]$ such that $f(x) = y$. In particular, $f(x) \in U$. But then $x \in f^{-1}(U) \subseteq U'$.

The proof of (4) is exactly the same as the proof of (3) if you replace all the Us with Vs. $\square$

Theorem 0.9. *Suppose that $a < b$. Show that $[a, b]$ is connected.*

Proof. Suppose that $[a, b]$ is not connected and let U, V be a disconnection. We will obtain a contradiction. Note first that either $a \in U$ or $a \in V$. Without loss of generality, we may assume that $a \in U$ (for if not, relabel U and V). Set S to be the set $\{x > a | [a, x) \subseteq U\}$. First let us make a few observations about the set S. Note that S is bounded above by any element of $V \cap [a, b]$ and such an element must since V is part of a disconnection of $[a, b]$. Therefore S has a least upper bound, call it L, note $L \leq b$. I claim that $(a, L) \subseteq U$. To see this claim, suppose not. Suppose that $z \in (a, L)$ and that $z \notin U$. Then $z \in V \cap [a, b]$ and so z is bigger than every element of S. So z is an upper bound for S. But $z < L$, contradicting the fact that L is a *least* upper bound. Now, $L \in [a, b] \subset U \cup V$, so there are two cases.

(1) $L \in U$. In this case, we can find a $d > 0$ such that $(L - d, L + d) \subseteq U$. But then $[a, L + d) = [a, L) \cup (L - d, L + d) \subseteq U$. In particular, $L + d \in S$. This contradicts the fact that L is an *upper bound* for S since $L + d > L$.

(2) $L \in V$. In this case, we can find a $d > 0$ such that $(L - d, L + d) \subseteq V$. Note $L - d > a$ since $a \notin V$. Choose $z = L - d/2$. This number is in V. Thus z is an upper bound for S. But note that $z < L$ which contradicts the fact that L is a *least* upper bound.

So in either case, we have a contradiction which completes the proof. $\square$

Theorem 0.10. *Prove that if $f : [a, b] \to \mathbb{R}$ is continuous, that the image of f is connected.*

Proof. Suppose not, then $D =$ image of f is disconnected. So $[a, b] = f^{-1}(D)$ is also disconnected by Proposition 0.8. But that contradicts Theorem 0.9. $\square$