

COMMENT ON THE LECTURE FROM WEDNESDAY SEPTEMBER 16TH

MATH 185-4
FALL 2009

At the end of the lecture, we were considering the following problem.

We had two possible limits for a function $f(x)$ as x goes to a and we were trying to deduce a contradiction. That is we assumed that

$$\lim_{x \rightarrow a} f(x) = L_1 \text{ and } \lim_{x \rightarrow a} f(x) = L_2.$$

From this, by choosing $\epsilon = |L_1 - L_2|/3$, we obtained that we showed that we could find an x such that

- $|f(x) - L_1| < \epsilon$ and
- $|f(x) - L_2| < \epsilon$.

Now we wanted to obtain a contradiction. I became slightly flummoxed in the last minute of class because I realized that the way I wanted to do this required a tool that we hadn't discussed yet in class: the *triangle inequality for real numbers*.

Theorem (The triangle inequality for real numbers). *For any two real numbers A and B , one has $|A + B| \leq |A| + |B|$.*

This is proven in the book (in an earlier chapter) and I had meant to prove it today but somehow I thought I didn't need it for this argument, and so I was planning to do it next.

However, assuming this inequality, let me show how you can give an algebraic proof of what we did in class. Adding the two equations, we get

$$|f(x) - L_1| + |f(x) - L_2| < 2\epsilon.$$

We can rewrite this as

$$|L_1 - f(x)| + |f(x) - L_2| < 2\epsilon.$$

(note that multiplying something inside an absolute value sign by -1 doesn't change anything). Then by the triangle inequality, (using $L_1 - f(x)$ for A and $f(x) - L_2$ for B , we get

$$|L_1 - f(x) + f(x) - L_2| \leq |L_1 - f(x)| + |f(x) - L_2| < 2\epsilon.$$

The left side of that string of inequalities is just $L_1 - L_2$ and so we see that

$$|L_1 - L_2| < 2\epsilon = 2|L_1 - L_2|/3$$

which is impossible ($\frac{2}{3}$ times a positive number does not make it bigger). This is what I intended to do, but realized I couldn't do (without the triangle inequality).

Of course, the argument, using the diagram to help illustrate, was also correct. It was just not algebraic.