EXTRA PROBLEMS # 3 – SOLUTIONS

Exercise 0.1. Show that $\sqrt{3}$ is not a rational number.

The proof can be based on the following fact: every integer $m \geq 2$ can be written uniquely written as $m = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1 < p_2 < \cdots < p_k$ are primes and $a_1, \ldots, a_k$ positive integers. Primes $p_1, \ldots, p_k$ are called as *prime factors* of m. Based on this fact you notice that every rational number q can be written as $q = m/n$, where m and n are integers, $n \geq 1$, and m and n do not have common prime factors.To prove that $\sqrt{3}$ is not a rational number, assume that it is and obtain a contradiction using the fact above.

Proof. Let us first observe that every integer m can be written in one of the three ways $m = 3k$, $m = 3k + 1$, or $m = 3k + 2$. (k is the integer part of $m/3$).

Let us now suppose towards a contradiction that $\sqrt{3}$ is rational, that is, $\sqrt{3} = m/n$ for some integers m and n, where $n > 0$. We can assume that m and n does not have common factors. Since $3 = (m/n)^2 = m^2/n^2$, we have

$$m^2 = 3n^2.$$

Thus m^3 is divisible by 3. We now show that $m = 3k$ for some k. Suppose that this is not the case but $m = 3k + 1$ or $m = 3k + 2$ for some k. Then

$$m^2 = (3k + 1)^2 = (3k)^2 + 6k + 1$$

or

$$m^2 = (3k + 2)^2 = (3k)^2 + 12k + 4.$$

Then m^2 is at the same time divisible by 3 and not divisible by 3. This is a contradiction, so we have $m = 3k$ for some k. However, this means that

$$3k^2 = n^2.$$

Thus, by applying the argument above for n, we observe that n is divisible by 3. So m and n both have factor 3. This is a contradiction, since we assumed that m and n does not have common factors. So $\sqrt{3}$ is not rational. □

Exercise 0.2. Let (a, b) be any open interval containing $\sqrt{3}$. Show that there exist rational numbers q and q' so that $a < q < \sqrt{3}$ and $\sqrt{3} < q' < b$.

Proof. Let $h > 0$ be the minimum of numbers $\sqrt{3} - a$ and $b - \sqrt{3}$. Fix an integer m so that $m > 1/h$. Let also k be the smallest integer so that $k > a/m$. Then $k/m > a$. Since $(k - 1)/m \leq a$ and $1/m < h$, we have that

$$\frac{k}{m} = \frac{k-1}{m} + \frac{1}{m} < a + h \leq a + (\sqrt{3} - a) = \sqrt{3}.$$

So $a < k/m < \sqrt{3}$ and we can take $q = k/m$.

Similarly, we can let n to be the smallest integer so that $n > \sqrt{3}/m$. Then $(n-1)/m \leq \sqrt{3}$ and

$$\frac{n}{m} = \frac{n-1}{m} + \frac{1}{m} < \sqrt{3} + h < b.$$

So we can take $q' = n/m$. $\square$

Exercise 0.3. Let (a, b) be any open interval containing $\sqrt{3}$. Show that there exist infinitely many rational numbers in (a, b).

Proof. By Problem 0.2, we know that there are rational numbers q and q' so that $a < q < \sqrt{3} < q' < b$. Let us denote $q_0 = q$ and $q'_0 = q'$. As (q_0, q'_0) is an interval containing $\sqrt{3}$ we can apply Problem 0.2 again to find rational numbers $q_0 < q_1 < \sqrt{3} < q'_1 < q'_0$. By repeating this argument we find any number n of rational numbers so that

$$a < q_0 < q_1 < q_2 < \cdots < q_n < \sqrt{3} < q'_n < \cdots < q'_1 < q'_0 < b.$$

As we can continue this process as long as we like and all the rational numbers found are contained in (a, b), we have that (a, b) contains infinitely many rational numbers. $\square$

Exercise 0.4. Show that all (non-empty) intervals contain infinitely many rational numbers.

Proof. Let I be an interval. Then I contains an (non-empty) open interval (a, b). So it suffices to show the claim only for an open interval (a, b).

Let $c = (a + b)/2$. By replacing $\sqrt{3}$ by c in the solution of Problem 0.2, we note that we can find rational numbers q and q' so that $a < q < c < q' < b$. Thus by replacing the use of Problem 0.2 in Problem 0.3 by this observation, we see that, for any n, (a, b) contains rational numbers

$$a < q_0 < q_1 < \cdots < q_n < c < q'_n < \cdots < q'_1 < q'_0 < b.$$

Thus (a, b) contains infinitely many rational numbers. $\square$

Exercise 0.5. Show that there exists a function f from $\mathbb{N}$ to $\mathbb{Z}$ so that f is onto.

Proof. Let $f\colon \mathbb{N} \to \mathbb{Z}$ be the function

$$f(n) = \begin{cases} n/2, & \text{if } n \text{ even or } n = 0 \\ -(n+1)/2, & \text{if } n \text{ odd.} \end{cases}$$

To show that f is onto, let $m \in \mathbb{Z}$. If $m \geq 0$, then $f(2m) = m$. If $m < 0$, then

$$f(-2m - 1) = -\frac{-2m - 1 + 1}{2} = -\frac{-2m}{2} = m.$$

$\square$