EXTRA PROBLEMS #6

SOLUTIONS

In this assignment, we give a proof for the chain rule. We gave an incomplete proof in class. In particular, we assumed that the inner-function, didn't actually achieve the value of its limit in a little interval around that point.

Consider the following function.

$$f(t) = \begin{cases} 0, & t = 0 \\ t^2 \sin(\frac{1}{t}), & t \neq 0 \end{cases}$$

We've talked about why this function is continuous in class before. Even more can be said however, this function is actually differentiable everywhere.

FACTS: We've done the following things in class (or done things close enough to them). You may use them without proving them.

(i) The functions

$$s(t) = \begin{cases} 0, & t = 0 \\ \sin(\frac{1}{t}), & t \neq 0 \end{cases} \text{ and } c(t) = \begin{cases} 0, & t = 0 \\ \cos(\frac{1}{t}), & t \neq 0 \end{cases}$$

are *NOT* continuous at $t = 0$. Even more, the functions $s(t)$ and $c(t)$ even have undefined limits at $t = 0$.

(ii) The functions $ts(t)$ and $tc(t)$ *ARE* continuous at $t = 0$. Here $ts(t)$ is just the product of the functions $i(t) = t$ with the function $s(t)$. We can also view the function $ts(t)$ as the multipart function

$$ts(t) = \begin{cases} 0, & t = 0 \\ t\sin(\frac{1}{t}), & t \neq 0 \end{cases}$$

Exercise 0.1. Prove that

$$f'(t) = \begin{cases} 0, & t = 0 \\ 2t\sin(\frac{1}{t}) - \cos(\frac{1}{t}), & t \neq 0 \end{cases}$$

Hint: The case where $t \neq 0$ should be easy, simply apply the chain and product rules. For the case of $t = 0$, you'll have to use the limit-definition of the derivative.

Proof. Let us consider the derivative at some t_0. If $t_0 \neq 0$, we can restrict our domain to $t \neq 0$, and then note that $f(t) = t^2 \sin(\frac{1}{t})$ is just made up of differentiable functions. Thus by the power and product rules, we have

$$f'(t_0) = 2t_0 \sin(\frac{1}{t_0}) + t_0^2 \cos(\frac{1}{t_0})(-1)(t_0^{-2}) = 2t_0 \sin(\frac{1}{t_0}) - \cos(\frac{1}{t_0})$$

at least for $t_0 \neq 0$. We now consider the case where $t_0 = 0$. Then

$$f'(t_0) = f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h)}{h} = \lim_{h \to 0} \frac{h^2 \sin(\frac{1}{h})}{h} = \lim_{h \to 0} h \sin(\frac{1}{h}) = hs(h).$$

But the function $hs(h)$ is continuous by fact (ii), and so the limit is equal to zero as desired. □

Exercise 0.2. Prove that the related function

$$k(t) = \begin{cases} 0, & t = 0 \\ 2t\sin(\frac{1}{t}), & t \neq 0 \end{cases}$$

is continuous but not differentiable.

Hint: You'll need to use the limit definition of the derivative to prove the function is not differentiable.

Proof. Note that $k(t) = 2ts(t)$, and so $k(t)$ is continuous since $ts(t)$ is continuous. To prove it is not differentiable, we need only prove it is not differentiable at a single point (k is actually differentiable at all $t \neq 0$). We will prove it is not differentiable at $t = 0$.

Suppose it was differentiable, then consider the following limit,

$$\lim_{h\to 0} \frac{k(0+h) - k(0)}{h} = \lim_{h\to 0} \frac{k(h)}{h} = \lim_{h\to 0} \frac{2h\sin(\frac{1}{h})}{h} = \lim_{h\to 0} 2\sin(\frac{1}{h}) = 2\lim_{h\to 0} s(h)$$

Now we know that the limit $\lim_{h\to 0} s(h)$ does not exist by fact (i) and so we are done. □

Exercise 0.3. Prove that $f'(t)$ is not continuous.

Hint: Show that $k(t)$ can be added to a non-continuous function, to get $f'(t)$.

Proof. Suppose that $f'(t)$ was continuous, but then we note that

$$f'(t) = \begin{cases} 0, & t = 0 \\ 2t\sin(\frac{1}{t}) - \cos(\frac{1}{t}), & t \neq 0 \end{cases} = k(t) - c(t).$$

Thus $c(t) = k(t) - f'(t)$. However, since $k(t)$ is continuous, and we are assuming that $f'(t)$ is continuous, this would imply that $c(t)$ is continuous. This contradicts fact (i), and so our assumption that $f'(t)$ was continuous must be incorrect. □

Now we get into the real work.

Theorem 0.4. *Let $f\colon (a,b) \to (c,d)$ and $g\colon (c,d) \to \mathbb{R}$ be differentiable functions. Then*

$$(g \circ f)'(x) = g'(f(x))f'(x)$$

for every $x \in (a,b)$.

We only proved this in class in the case that f did not do exactly what the f function above did (equal its limit many times around the limiting value). We begin with a warm-up exercise that gives us a new perspective to the derivative.

Exercise 0.5. Suppose $f\colon (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$ and set $L = f'(x_0)$. Consider a new function $E\colon (-\delta,\delta) \setminus \{0\} \to \mathbb{R}$ defined by the formula

$$E(h) = \frac{f(x_0+h) - f(x_0)}{h} - L.$$

Here $\delta > 0$ is assumed to be chosen in such a way that E can be defined.

Show that

a. for every $h \in (-\delta,\delta) \setminus \{0\}$ we have

$$f(x_0+h) = f(x_0) + Lh + hE(h).$$

b. $\lim_{h\to 0} E(h) = 0$.

Proof. We first show part (a). We simply plug in the definition of $E(h)$ to obtain

$$f(x_0) + Lh + hE(h) = f(x_0) + Lh + h\left(\tfrac{f(x_0+h)-f(x_0)}{h} - L\right) = f(x_0) + Lh - Lh + f(x_0+h) - f(x_0) = f(x_0+h)$$

which is exactly what we want.

To prove (b) we note that $L = f'(x_0)$ is a constant and observe that

$$\lim_{h\to 0} E(h) = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h} - L\right) = \lim_{h\to 0}\left(\frac{f(x_0+h)-f(x_0)}{h}\right) - \lim_{h\to 0} L = f'(x_0) - L = L - L = 0$$

as desired. □

The function E defined in the exercise can be called as an *error term*. This error term actually characterizes the derivative as the following exercise shows.

Exercise 0.6. Let $f\colon (a,b)\to\mathbb{R}$ be a function and $x_0\in(a,b)$. Suppose that there exists a function $E\colon(-\delta,\delta)\setminus\{0\}\to\mathbb{R}$ so that $\lim_{h\to 0}E(h) = 0$ and

$$f(x_0+h) = f(x_0) + Lh + hE(h)$$

where L is a number. Show that f is differentiable at x_0 and $f'(x_0) = L$.

Proof. Consider $\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$. We plug in the equality from the exercise and we get

$$\lim_{h\to 0}\frac{f(x_0)+Lh+hE(h)-f(x_0)}{h} = \lim_{h\to 0}\frac{Lh+hE(h)}{h} = \lim_{h\to 0}(L+E(h)) = L + 0 = L = f'(x_0)$$

□

These two exercises can be combined as a *lemma* which we will use later.

Lemma 0.7. *A function $f\colon(a,b)\to\mathbb{R}$ has derivative L at x_0 if and only if there exist $\delta>0$ and $E_f\colon(-\delta,\delta)\setminus\{0\}\to\mathbb{R}$ so that*

(1) $\lim_{h\to 0}E_f(h) = 0$, *and*
(2) $f(x_0+h) = f(x_0) + Lh + hE_f(h)$

for every $0<|h|<\delta$.

Exercise 0.8. Let $f\colon(a,b)\to(c,d)$ and $g\colon(c,d)\to\mathbb{R}$ be functions and $x_0\in(a,b)$. Suppose that f is differentiable at x_0 and g is differentiable at $f(x_0)$. Let $E_f\colon(-\delta_0,\delta_0)\to\mathbb{R}$ be an error term for f (here we define $E_f(0)=0$) and let $E_g\colon(-\delta_1,\delta_1)\to\mathbb{R}$ be an error term for g (we also define $E_g(0)=0$). Let also $\ell_f\colon(-\delta_0,\delta_0)\to\mathbb{R}$ be the function $\ell_f(h) = f'(x_0)h + hE_f(h)$.

a. Show that there exists $\delta>0$ so that for every $h\in(-\delta,\delta)$ we have

$$|\ell_f(h)| < \delta_1.$$

(*Hint:* Use the limit $\lim_{h\to 0}\ell_f(h)$)

b. Show that with this δ we have

$$\begin{aligned} g(f(x_0+h)) &= g\left(f(x_0)+\ell_f(h)\right)\\ &= g(f(x_0)) + g'(f(x_0))\ell_f(h) + \ell_f(h)E_g\left(\ell_f(h)\right)\end{aligned}$$

for every $0<|h|<\delta$.

c. Show that there exists a function $E\colon(-\delta,\delta)\setminus\{0\}\to\mathbb{R}$ so that

$$g'(f(x_0))\ell_f(h) + \ell_f(h)E_g\left(\ell_f(h)\right) = g'(f(x_0))f'(x_0)h + hE(h)$$

for every $0<|h|<\delta$ and $\lim_{h\to 0}E(h) = 0$.

d. Prove Theorem 0.4.

Proof. (a) We first show that $\lim_{h\to 0}\ell_f(h) = 0$, but $\lim_{h\to 0}\ell_f(h) = \lim_{h\to 0} = f'(x_0)h + hE_f(h) = f'(x_0)(0) + (0)(0) = 0 + 0 = 0$. Consider δ_1 as an epsilon for this limit, thus there exists a δ such that for all $h \neq 0$, $h \in (-\delta, \delta)$, we have $\ell_f(h) \in (-\delta_1, \delta_1)$ and this condition is exactly the same as our desired conclusion that $|\ell_f(h)| < \delta_1$.

(b) We have two equalities to prove. First note that

$$g(f(x_0+h)) = g(f(x_0) + Lh + hE_f(h)) = g(f(x_0) + \ell_f(h))$$

so the first equality is easy. For the second equality, set $y_0 = f(x_0)$ and set $z = \ell_f(h)$. Then $g\left(f(x_0) + \ell_f(h)\right) = g(y_0 + z)$ and since g is differentiable at y_0 and $z \in (-\delta_1, \delta_2)$, we have

$$g(y_0+z) = g(y_0) + g'(y_0)z + zE_g(z) = g(f(x_0)) + g'(f(x_0))\ell_f(h) + \ell_f(h)E_g(\ell_f(h))$$

as desired.

(c) We simply note that

$$g'(f(x_0))\ell_f(h)+\ell_f(h)E_g\left(\ell_f(h)\right) = g'(f(x_0))(f'(x_0)h+hE_f(h))+(f'(x_0)h+hE_f(h))E_g\left(\ell_f(h)\right)$$

by distributing and factoring, we get that the right side is equal to

$$g'(f(x_0))f'(x_0)h + h(g'(f(x_0))E_f(h) + (f'(x_0) + E_f(h))E_g\left(\ell_f(h)\right))$$

so we define $E(h) = g'(f(x_0))E_f(h) + (f'(x_0) + E_f(h))E_g\left(\ell_f(h)\right)$. We simply need to show that $\lim_{h\to 0} E(h) = 0$. But this follows easily since $\lim_{h\to 0} E_f(h) = 0$ and

$$\lim_{h\to 0} E_g\left(\ell_f(h)\right) = E_g\left(\lim_{h\to 0}\ell_f(h)\right) = E_g(0) = 0.$$

We can pull the limit inside because E_g is defined so as to be continuous at zero.

(d) Combining exercise (b) and (c) we obtain that

$$g(f(x_0+h)) = g(f(x_0)) + g'(f(x_0))f'(x_0)h + hE(h)$$

and applying the previous lemma (for $L = g'(f(x_0))f'(x_0)$) completes the proof. □

The formula

(1) $$f(x_0+h) = f(x_0) + f'(x_0)h + hE(h)$$

can also be used to approximate the values of function f.

Exercise 0.9. Argue how (1) could be used to give a decimal approximation for $\sqrt{25.012}$ if you are able to assume that the error term in (1) is very small. Calculate an approximation using (1) and compare it to a result given by a calculator.

Proof. We know that $\sqrt{x}$ is differentiable at $x = 25$. Thus, we can write $\sqrt{x_0+h} = \sqrt{x_0} + (\frac{1}{2})x_0^{\frac{-1}{2}}h + hE(h)$ as above. Since $E(h)$ goes to zero as h does, $hE(h)$ goes to zero faster than $(\frac{1}{2})x_0^{\frac{-1}{2}}h$, which is just a constant times h (one can make this precise if one wants, I will not do this here). This means that we can approximate $\sqrt{x_0+h}$ by $\sqrt{x_0} + (\frac{1}{2})x_0^{\frac{-1}{2}}h$. Now, plugging in $x_0 = 25$ and $h = 0.012$ we get

$$\sqrt{25.012} \sim \sqrt{25} + (\frac{1}{2})25^{\frac{-1}{2}}(0.012) = 5 + (\frac{1}{10})(0.012) = 5.0012$$

On the other hand, using a calculator, one sees that $\sqrt{25.012} = 5.001199856\ldots$. □